# Capacitor and electric capacitance. Energy in capacitors. Magnetic field. Inductor and self-inductance. Energy in inductors.

## 3.1. Capacitors

### 3.1.1. Capacitor and capacitance

Using the electrostatic phenomena, it is possible to define a new two-terminal element, called capacitor. The capacitor consists of two conductive parallel plates with a dielectric between them (fig. 3.1). When a voltage difference $v_C$ is applied on them, the charged particles cannot pass through the dielectric, so the positive electric charges $Q^+$ are stored on one of the plates and the negative charges $Q^-$ - on the other one. The ratio between $Q$ and $v_C$ is called electric capacity:

$$C=\frac{Q}{v_C}$$

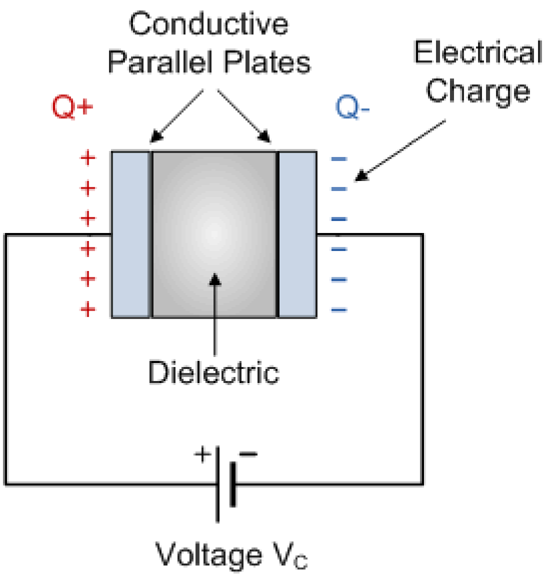


Fig. 3.1.

The unit for capacity is Farad [F], however more often are used the smaller units:

- $1\,mF=10^{-3}\,F$
- $1\,\mu F=10^{-6}\,F$
- $1\,nF=10^{-9}\,F$
- $1\,pF=10^{-12}\,F$

The electric symbol for capacitor is presented in fig. 3.2.

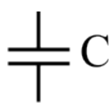


Fig. 3.2. Symbol for capacitor.

### 2.1.2. Current and voltage of capacitors

The current flow of the capacitor could be estimated from the equation for electric current:

$$i_C(t)=\frac{dQ}{dt}$$

By substituting $Q=C.v_C$ (from $C=\frac{Q}{v_C}$¿ , the above equation becomes:

$$i_C(t)=\frac{dQ}{dt}=\frac{d(C.v_C)}{dt}=C\frac{d v_C}{dt}+v_C\frac{dC}{dt}$$

If the electric capacity is constant ( $C=const$ ), the current of the capacitor becomes:

$$i_C(t)=C\frac{d v_C}{dt}+v_C\frac{dC}{dt}=C\frac{d v_C}{dt}$$

The voltage drop on the capacitor could be derived by integrating the above equation:

$$v_C(t)=\frac{1}{C}\int_0^t i_C(t)\,dt+v_C(0)$$

**Example:** What is the voltage drop on the capacitor for the DC circuit in fig. 3.3.

Considering the current equation $i_C(t)=C\frac{d v_C}{dt}$ if the capacitor voltage is constant ( $v_C=const$ ), no current will flow through it:

$$i_C=0\,A$$

The KVL for the circuit is:

$$V_{SRC}=V_R+v_C=i_C.R+v_C=0+v_C$$

Then the voltage drop on the capacitor is equal to the applied voltage:

$$v_C=V_{SRC}=10\,V$$

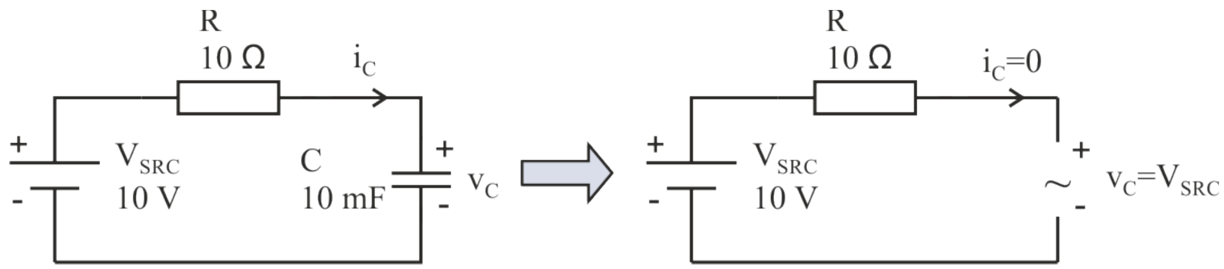


Fig. 3.3.

**Example:** What is the voltage drop on the capacitor for the DC circuit in fig. 3.4.
The current of the capacitor is once again equal to 0:

$$i_C=0\,A$$

The KCL for the circuit is:

$$i=i_C+i_R=0+i_R=i_R$$

By applying Ohm's law the current $i_R$ becomes:

$$I_R=\frac{V_{SRC}}{R}=\frac{6}{20}=0.3\,A$$

The resistor and the capacitor are connected in parallel, which means their voltages are equal:

$$v_C=v_R=R.i_R=20*0.3=6\,V$$

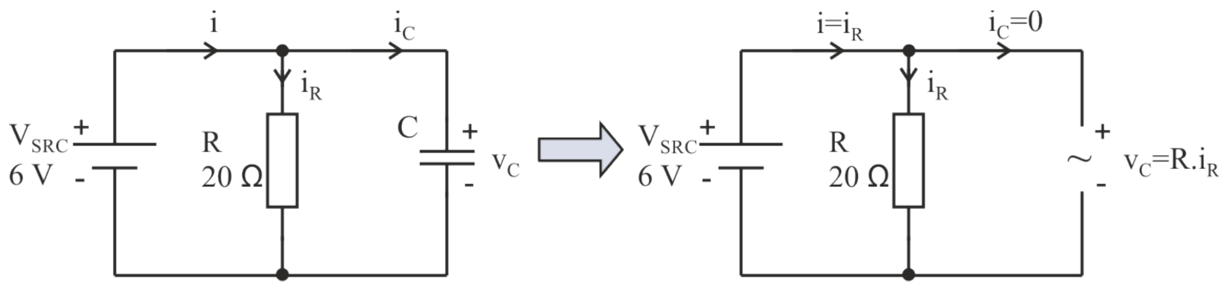


Fig. 3.4.

### 3.1.3. Capacitors in series

Consider the circuit from fig. 3.5 with three capacitors connected in series. According to KVL the cumulative voltage drop $v$ on the three capacitors is:

$$v=v_{C1}+v_{C2}+v_{C3}=\frac{1}{C_1}\int_0^t i_C\,dt+\frac{1}{C_2}\int_0^t i_C\,dt+\frac{1}{C_3}\int_0^t i_C\,dt=\left(\frac{1}{C_1}+\frac{1}{C_2}+\frac{1}{C_3}\right)\int_0^t i_C\,dt=\frac{1}{C_E}\int_0^t i_C\,dt$$

Then the equal capacitance is:

$$C_E=\frac{1}{\frac{1}{C_1}+\frac{1}{C_2}+\frac{1}{C_3}}$$

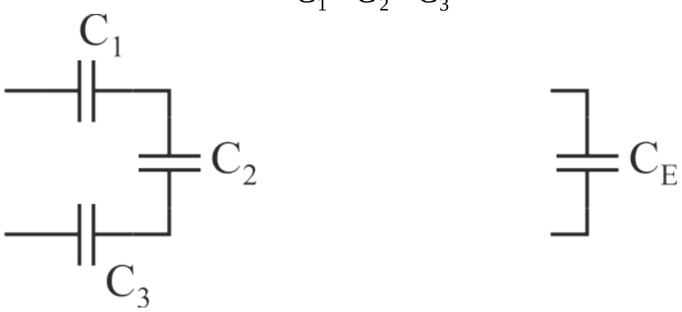


Fig. 3.5.

### 3.1.4. Capacitors in parallel

Consider the circuit in fig. 3.6 with three capacitors connected in parallel. The KCL for the circuit is:

$$i=i_1+i_2+i_3=C_1\frac{d v_C}{dt}+C_2\frac{d v_C}{dt}+C_3\frac{d v_C}{dt}=\left(C_1+C_2+C_3\right)\frac{d v_C}{dt}=C_E\frac{d v_C}{dt}$$

Then the equivalent capacitance is:

$$C_E=C_1+C_2+C_3$$

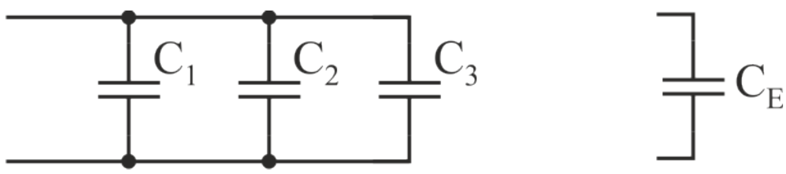


Fig. 3.6.

### 3.1.5. Energy stored in capacitors

Consider a situation where a capacitor is connected to a DC voltage source $V_{SRC}$ (fig. 3.7). Under its influence the two plates of the capacitor are charged $+Q$ and $-Q$ respectively. This leads to the creation of an electric field between the two plates, acting on the dielectric. Since the electric field could move charges (do work), this means the capacitor is charged with a certain energy.

The power entering the capacitor when connected to the source $V_{SRC}$ is:

$$p_C(t)=i_C(t).v_C(t)=C\frac{d v_C}{dt}v_C$$

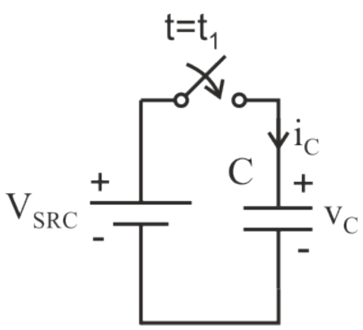


Fig. 3.7.

If the capacitor is connected to $V_{SRC}$ in the moment of time $t=t_1$ and in the moment of time $t=t_2$ the capacitor is fully charged, then the charged energy $W_C$ could be estimated by integrating the power $p_C(t)$ from $t_1$ to $t_2$ :

$$W_C=\int_{t_1}^{t_2} p_C(t)\,dt=\int_{t_1}^{t_2} C.v_C.\frac{d\,v_C}{dt}dt=\int_{0}^{V_{SRC}} C.v_C.d\,v_C=\frac{1}{2}.C.v_C^{\,2}$$

In other words the maximal energy which could be stored in a capacitor in the form of electric field is:

$$W_C=\frac{1}{2}.C.v_C^{\,2}$$

**Example:** A capacitor $C=10\,mF$ is connected to a DC voltage source $V=100\,V$ . What energy will be stored in the capacitor?

$$W_C=\frac{1}{2}.C.v_C^{\,2}=\frac{1}{2}.10.10^{-3}.100^2=50\,J$$

## 3.2. Magnetic field and inductors

### 3.2.1. Magnetic field and basic quantities

A magnetic field appears near moving electric charges as well as around alternating electric field. The magnetic field is characterized with a magnetic induction $\vec{B}$ (often called simply magnetic field). The force $\vec{F_M}$ which acts on a charge $q$ , moving with speed $\vec{v}$ , is (fig. 3.8):

$$\vec{F_M}=q.(\vec{v}\times\vec{B})$$

The magnetic field $\vec{B}$ can also be defined using the magnetic force $\vec{F_M}$ acting on an conductor with length $\vec{l}$ , with current flow $I$ :

$$\vec{F_M}=I.(\vec{l}\times\vec{B})$$

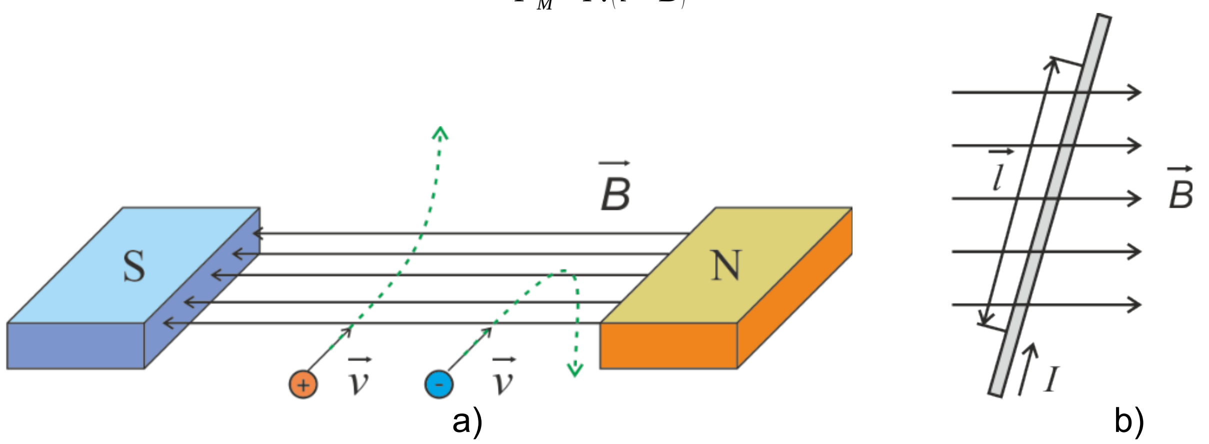


Fig. 3.8. Magnetic force acting on a moving charge (a) and acting on a conductor (b).

If $\vec{l}$ and $\vec{B}$ are orthogonal, then the magnetic induction is:

$$B=\frac{F_M}{I.l}$$

The unit for magnetic induction is Tesla [T].
Consider a permanent magnet, which creates magnetic field $\vec{B}$ (fig. 3.9).

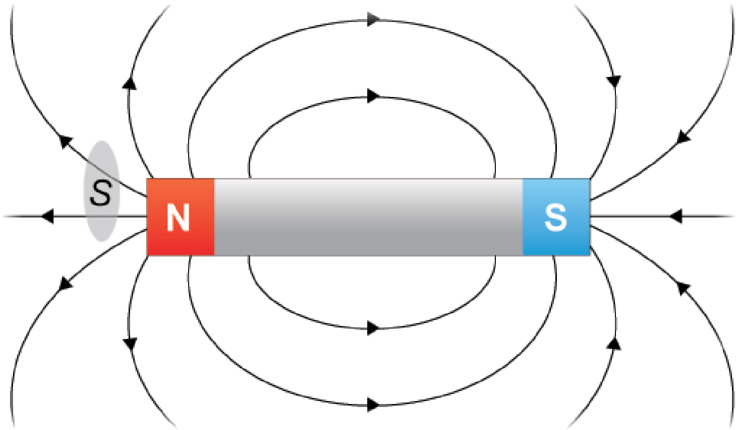


Fig. 3.9.

The integral of the magnetic field over the surface $\vec{S}$ is called magnetic flux $\Phi$ :

$$\Phi=\int_{(S)} \vec{B}\, d\vec{S}$$

The unit for magnetic flux is Weber [Wb].
If $\vec{S}$ is a closed surface (for example the surface of a sphere surrounding the magnet) then the magnetic flux is null (fig. 3.10):

$$\Phi=\oint_{(S)} \vec{B}\, d\vec{S}=0$$

The above equation is Gauss's law for magnetism and is also one of Maxwell's equations. This law is a consequence of the assumption that magnetic monopoles do not exist.

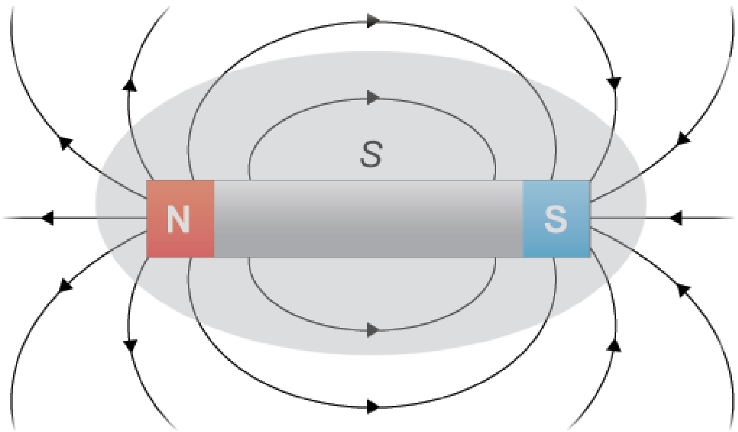


Fig. 3.10.

### 3.2.2. Inductors and self-inductance

When a current $i$ flows in a conductor wire, a magnetic field $\vec{B}$ is created around it, with direction according to the right-hand rule – if the thumb points in the direction of the current, the direction of the magnetic field can be found by curving one's fingers around the wire (2.11a). The circulation of the magnetic field over a closed loop $\vec{l}$ is given with:

$$\oint_{(l)} \vec{B} d\vec{l} = \mu . i$$

where $\mu$ is the magnetic permeability of the medium.
In case the wire has multiple turns $N$ then the equation becomes:

$$\oint_{(l)} \vec{B} d\vec{l} = N . \mu . i$$

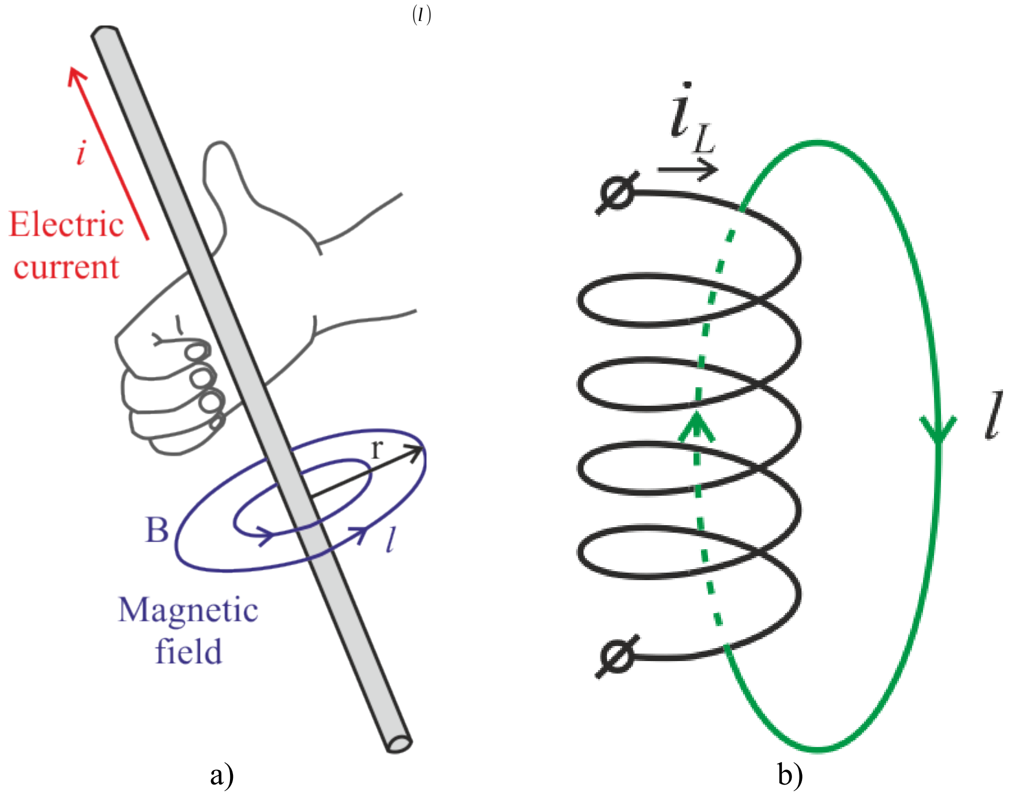


Fig. 3.11.

The ratio between magnetic flux $\Phi$ and the current $i$ , which is creating it, is given with the coefficient of self inductance (or simply inductance):

$$L = \frac{\Phi}{i}$$

$L$ depends on the magnetic permeability of the medium and the form and contour of the wire.
The unit for inductance is Henry [H].
In case the wire has multiple turns $N$ , the above equation becomes:

$$L = N . \frac{\Phi}{i}$$

This allows to define a new two-terminal element called inductor (also called coil), characterized by an inductance $L$ . The coil has multiple windings or turns (fig. 3.12a) and is used to store energy in the form of magnetic field. The electric symbol for inductor is shown in fig. 3.12b.

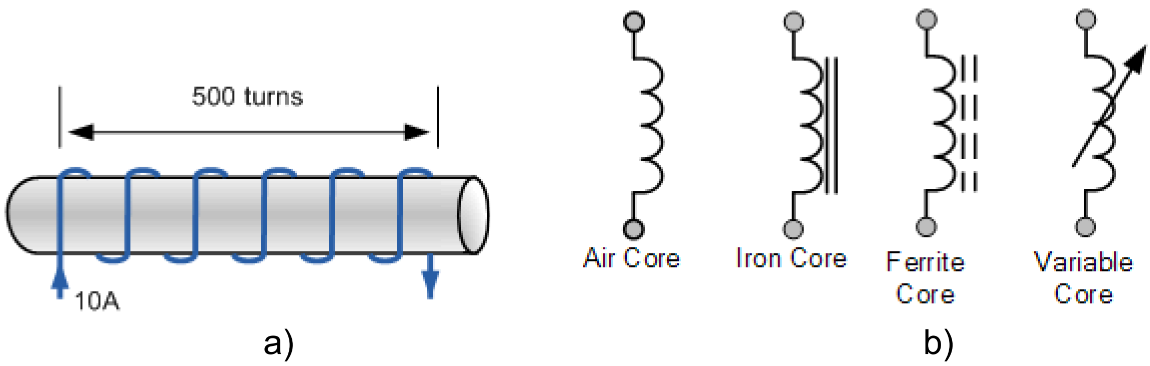

Fig. 3.12. An inductor (a) and the electric symbol for inductor (b).

### 3.2.3. Faraday's law of induction and mutual inductance

Faraday's law states that any change in the magnetic flux trough a closed wire loop will produce an electromotive force (emf or voltage) in the loop, which would create a current $i$ (фиг. 2.13):

$$e = \frac{-d\Phi}{dt}$$

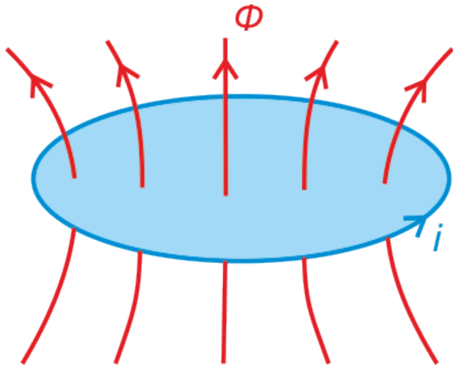


Fig. 3.13.

In case the wire has $N$ turns, the equation becomes:

$$e = -N.\frac{d\Phi}{dt}$$

The minus (-) sign was in fact added later by Heinrich Lenz and is called Lent's law. It states that when an emf is generated by a change in the magnetic flux, the polarity of the produced emf is such that it produces a current whose magnetic field will oppose the change which produces it (fig. 3.14). In other words the loop tries to maintain the magnetic flux through it by producing emf which compensates the change $\frac{d\Phi}{dt}$ .

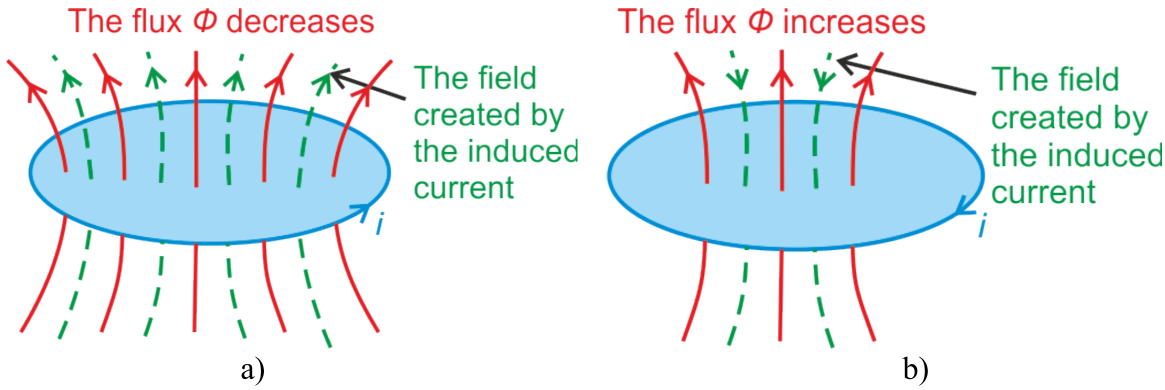


Fig. 3.14. Closed loop responses to changes in the magnetic flux: a) the flux is decreasing; b) the flux is increasing.

Consider an inductor $L_1$ over which flows a current $i_1$ , which creates a magnetic flux $\Phi_1$ . A part of this magnetic flux ( $\Phi_{12}$ ) reaches a second inductor $L_2$ which produces emf (fig. 3.15.). The ratio between $\Phi_{12}$ and the current which creates it is called mutual inductance:

$$M = \frac{\Phi_{12}}{i_1}$$

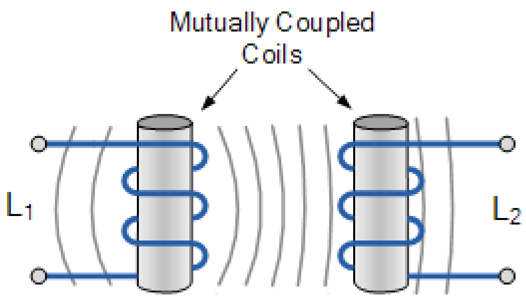


Fig. 3.15. Mutual inductance

### 3.2.4. Current and voltage of inductors

The voltage drop on an inductor could be estimated from Faraday's law. The voltage drop $v_L$ has the opposite direction of the emf:

$$v_L(t)=-e=\frac{d\Phi}{dt}$$

Considering the magnetic flux is $\Phi=L.i_L$ , the above equation becomes:

$$v_L(t)=\frac{d\Phi}{dt}=\frac{d(L.i_L)}{dt}=L.\frac{di_L}{dt}+i_L.\frac{dL}{dt}$$

If the inductance is a constant ( $L=const$ ), then the voltage drop on an inductor is:

$$v_L(t)=L.\frac{di_L(t)}{dt}$$

By integrating the above equation could be derived the current of an inductor:

$$i_L(t)=\frac{1}{L}\int_0^t v_L(t)dt+i_L(0)$$

These equations show that there is no voltage drop on the inductor when there are no changes in the current flow. In other words the resistance of an ideal inductor in DC circuits is 0.

**Example:** What is the current of the coil for the DC circuit in fig. 3.16?

Considering this is a DC circuit, there is no voltage drop on the resistor which means it is a short circuit. The KVL for this circuit is:

$$V_{SRC}=V_R+V_L=V_R+0=V_R$$

Then the current through the inductor becomes:

$$I_L=\frac{V_{SRC}}{R}=\frac{10}{10}=1A$$

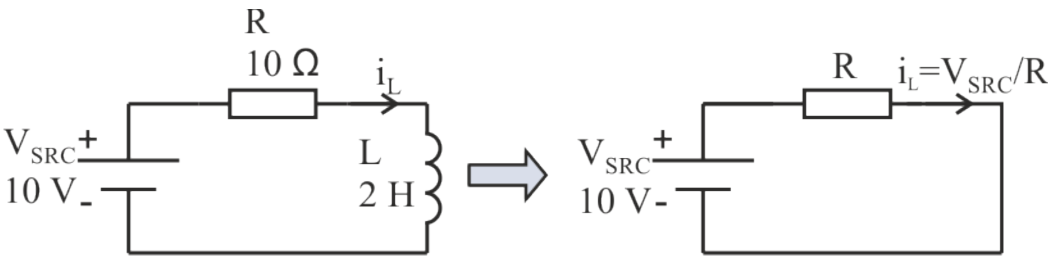


Fig. 3.16.

**Example:** What is the current through the inductor for the DC circuit in fig. 3.17?

Considering this is a DC circuit, the inductor has resistance 0. Then the equivalent resistance of $R$ and $L$ is:

$$R_E=\frac{R.0}{R+0}=0$$

In other words the inductor short circuits the resistor $R$ and no current flows through it.

The KCL for the circuit below is:

$$i=i_R+i_L=0+i_L=i_L$$

Then by writing Ohm's law the current is:

$$i_L = \frac{V_{SRC}}{R_0} = \frac{10}{10} = 1\,A$$

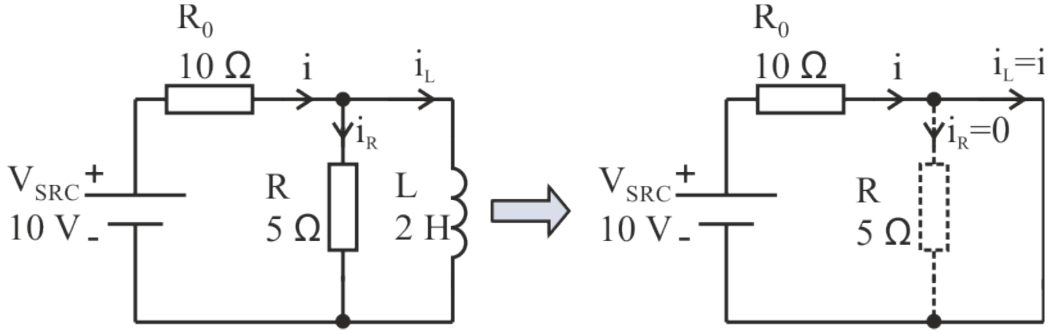


Fig. 3.17.

### 3.2.5. Inductors in series

Consider the circuit from fig. 3.18 with three inductors connected in series. According to KVL the cumulative voltage drop $v$ on the three inductors is:

$$v = v_1 + v_2 + v_3 = L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + L_3 \frac{di}{dt} = \left( L_1 + L_2 + L_3 \right) \frac{di}{dt} = L_E \frac{di}{dt}$$

Then the equal inductance is:

$$L_E = L_1 + L_2 + L_3$$

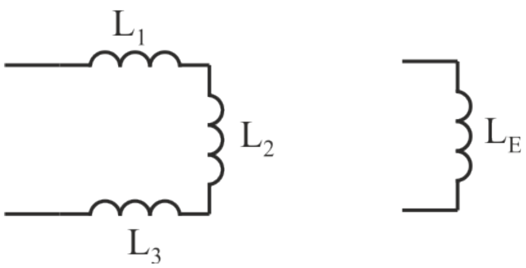


Fig. 3.18.

### 3.2.6. Inductors in parallel

Consider the circuit from fig. 3.19 with three inductors connected in parallel. According to KCL the entering current is:

$$i = i_1 + i_2 + i_3 = \frac{1}{L_1} \int_0^t v_L\, dt + \frac{1}{L_2} \int_0^t v_L\, dt + \frac{1}{L_3} \int_0^t v_L\, dt = \text{¿}$$

$$\text{¿} \left( \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} \right) \int_0^t v_L\, dt = \frac{1}{L_E} \int_0^t v_L\, dt$$

Then the equal inductance is:

$$L_E = \frac{1}{\frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}}$$

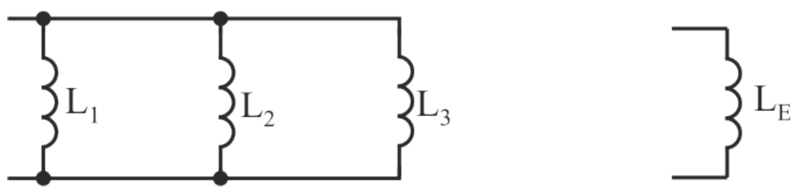


Fig. 3.19.

### 3.2.7. Energy stored in inductors

Consider a situation where an inductor is connected to a DC voltage source $V_{SRC}$ through a resistor $R$ (fig. 3.20). The current which start flowing in the inductor produces a magnetic field, which is able to move electric charges. This means the inductor stores energy in the form of magnetic field.

The DC current which will flow in the circuit is:

$$I=\frac{V_{SRC}}{R}$$

The power entering the inductor when connected to the source is:

$$p_L(t)=i_L(t).v_L(t)=L\frac{di_L}{dt}i_L$$

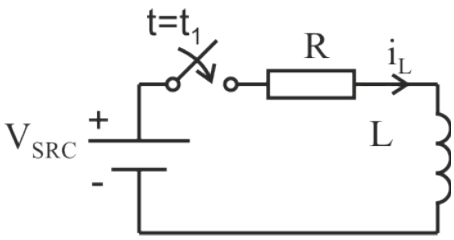


Fig. 3.20.

If the inductor is connected to the source in the moment of time $t=t_1$ and in the moment of time $t=t_2$ the inductor is fully charged, then the charged energy $W_L$ could be estimated by integrating the power $p_L(t)$ from $t_1$ to $t_2$ :

$$W_L=\int_{t_1}^{t_2} p_L(t)dt=\int_{t_1}^{t_2} L.i_L.\frac{di_L}{dt}dt=\int_0^I L.i_L.di_L=\frac{1}{2}.L.i_L^2$$

In other words the maximal energy which could be stored in an inductor in the form of magnetic field is dependent on the inductance and the current flow:

$$W_L=\frac{1}{2}.L.i_L^2$$

**Example:** A current of $I=10A$ is flowing through an inductor $L=20mH$ . What energy is stored in the inductor?

$$W_L=\frac{1}{2}.CL.i_L^2=\frac{1}{2}.20.10^{-3}.10^2=1J$$

### References

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3. http://www.electronics-tutorials.ws/category/capacitor
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