NONSINUSOIDAL STEADY STATE CIRCUIT ANALYSIS

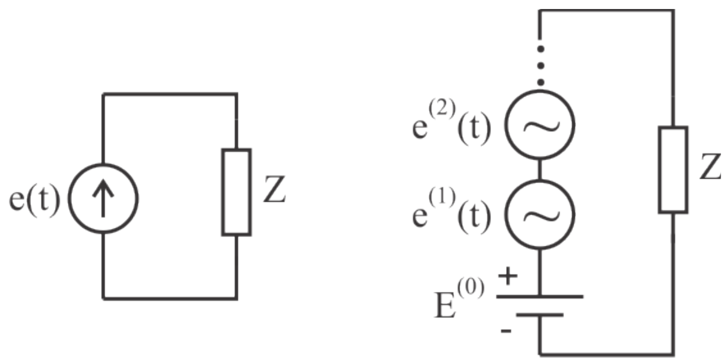
1. Introduction

Every nonsinusoidal signal can be decomposed into a Fourier series as a sum of sinusoidal components:

$$e(t)=E^{(0)}+E_m^{(1)}\sin\left(\omega t+\varphi^{(1)}\right)+E_m^{(2)}\sin\left(2\omega t+\varphi^{(2)}\right)+\ldots=E^{(0)}+e^{(1)}(t)+e^{(2)}(t)+\ldots$$

where $E^{(0)}$ is the DC offset, $E^{(k)}$ is the amplitude of the kth harmonic component, and $\varphi^{(k)}$ – is the phase shift of the kth harmonic component.

This means we can substitute the nonsinusoidal voltage source $e(t)=E^{(0)}+e^{(1)}(t)+e^{(2)}(t)+\ldots$ with numerous sinusoidal voltage sources ($E^{(0)}$, $e^{(1)}(t)$, $e^{(2)}(t)$, etc.), connected in series. And since we are investigating linear circuits, the superposition theorem could be applied so that the circuit is solved for each source independently.



Let the inductors and capacitors are defined with their inductances L and capacitances C . Their complex impedances would be different for each harmonic component, which means that $Z=R+j\left(X_L-X_C\right)$ will have different value for each component.

For frequency f the reactances are:

$$X_L=2\pi fL=\omega L \qquad X_C=\frac{1}{2\pi fC}=\frac{1}{\omega C}$$

for frequency $2f$ they are:

$$X_L=2\pi(2f)L=2\omega L \qquad X_C=\frac{1}{2\pi(2f)C}=\frac{1}{2\omega C}$$

for frequency $3f$ they are:

$$X_L=2\pi(3f)L=3\omega L \qquad X_C=\frac{1}{2\pi(3f)C}=\frac{1}{3\omega C}$$

and so on. The RMS value of the nonsinusoidal currents and voltages are estimated with:

$$U=\sqrt{\left(U^{(0)}\right)^2+\left(\frac{U_m^{(1)}}{\sqrt{2}}\right)^2+\left(\frac{U_m^{(2)}}{\sqrt{2}}\right)^2+\ldots}=\sqrt{\left(U^{(0)}\right)^2+\frac{\left(U_m^{(1)}\right)^2+\left(U_m^{(1)}\right)^2+\ldots}{2}}$$

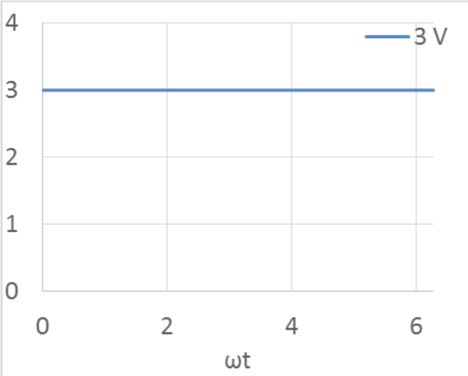
and and active power with:

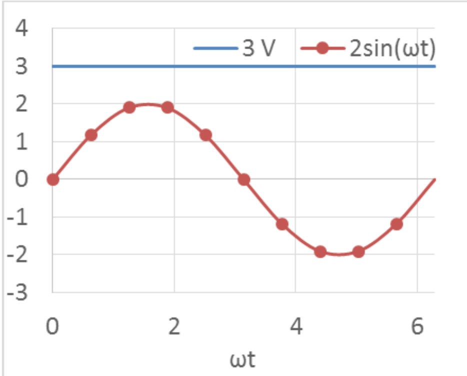
$$P=P^{(0)}+P^{(1)}+P^{(2)}+\ldots$$

2. Problems

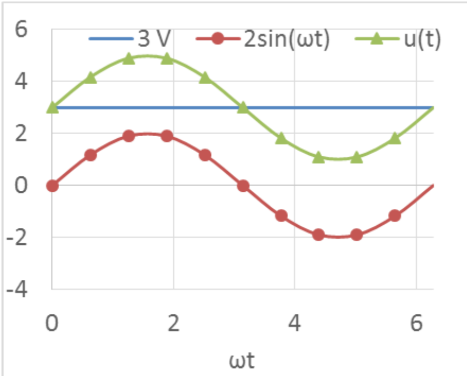
Problem 1. Sketch the nonsinusoidal voltage $u(t)=3+2\sin\omega t[V]$.

Solution: First we are going to sketch the DC component $3[V]$, and then we'll add the sinusoidal component $2\sin\omega t[V]$.





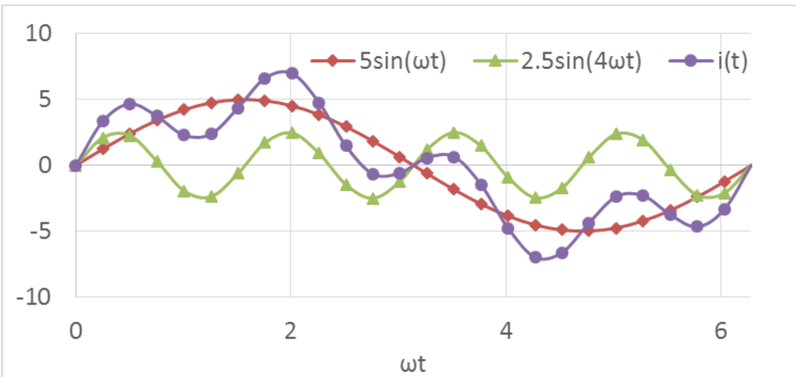
Finally we are going to sum them graphically, in order to obtain $u(t)$:



It can be see that as a result $2\sin\omega t$ varies around $3V$, instead of $0V$.

Problem 2. Sketch the nonsinuisoidal current $i(t)=5\sin\omega t+2{,}5\sin 4\omega t$

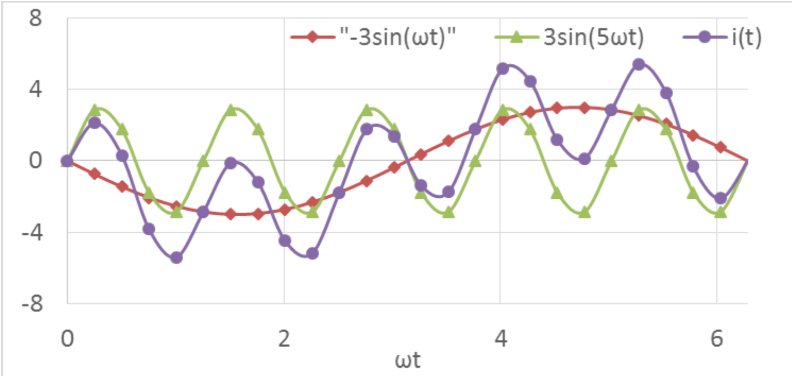
Solution: First we will sketch the fundamental component $5\sin\omega t$, then we'll add the 4[th] harmonic $2{,}5\sin 4\omega t$, and finally we will add sum them graphically.



It can be seen that as a result the 4[th] harmonic component $2{,}5\sin 4\omega t$ varies around the fundamental one $5\sin\omega t$, instead of varying around $0A$.

Problem 3. Sketch the following nonsinusoidal waveform $i(t)=-3\sin\omega t+3\sin 5\omega t$.

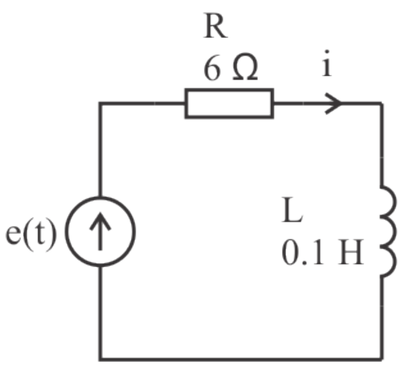
Solution: First we will sketch the fundamental component $-3\sin\omega t$, then we'll add the 5th harmonic $3\sin 5\omega t$ and at the end we will sum them graphically.

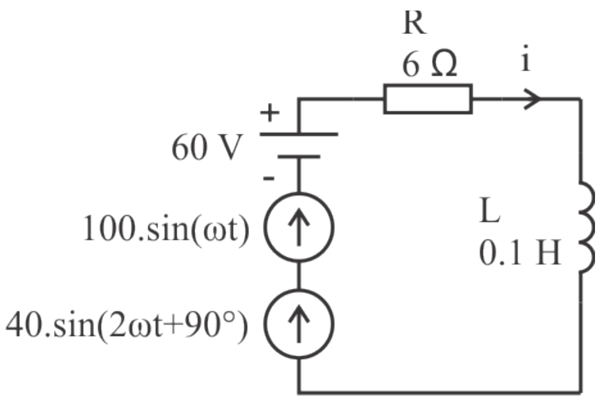


Problem 4. The circuit below is powered by a nonsinusoidal voltage source $e(t)=60+100\sin(200t)+40\sin(400t+90°)V$:

a) For each of the harmonic components obtain the reactances, create an equivalent circuit with complex numbers and estimate the current i ;

b) Obtain the instantaneous and RMS currents $i(t)=?\,I=?$ and the power dissipated by the resistor $P_R=?$

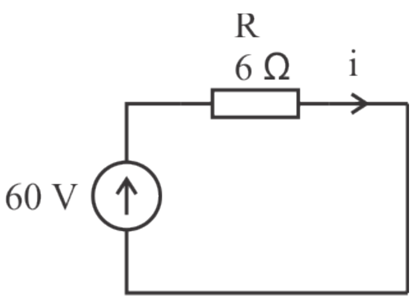




Solution a) The circuit is linear so we apply the superposition theorem and solve the circuit for each source individually.

Only the DC component is acting $E^{(0)}=60[V]$

In DC circuits the inductors are a short circuit, so the equivalent circuit is:



The current is:

$$I^{(0)}=\frac{60}{6}=10[A]$$

<u>Only the fundamental harmonic is acting</u> $e^{(1)}(t)=100\sin(200t)[V]$

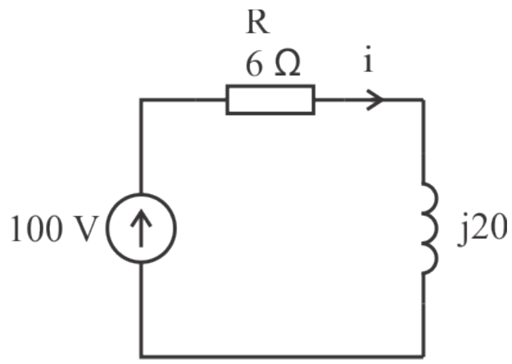
The angular frequency is $\omega=200[rad/s]$, so the inductive reactance is:

$$X_L=\omega L=200.0{,}1=20[\Omega]$$

We present the voltage source as phasor:

$$e^{(1)}(t)=100\sin(200t)\rightarrow \dot{E}^{(1)}=100\,e^{j0°}=100[V]$$

The equivalent circuit with complex numbers is:



The complex current is obtained from the KVL:

$$\dot{I}=\frac{100}{6+j20}=\frac{100}{\sqrt{6^2+20^2}\,e^{j\,atan\frac{20}{6}}}=\frac{100}{20{,}88\,e^{j73°}}=4{,}79\,e^{-j73°}[A]$$

In sinusoidal form:

$$\dot{I}=4{,}79\,e^{-j73°}\rightarrow i^{(1)}(t)=4{,}79\sin(\omega t-73°)[A]$$

<u>Only the second harmonic component is acting</u> $e^{(2)}(t)=40\sin(400t+90°)[V]$

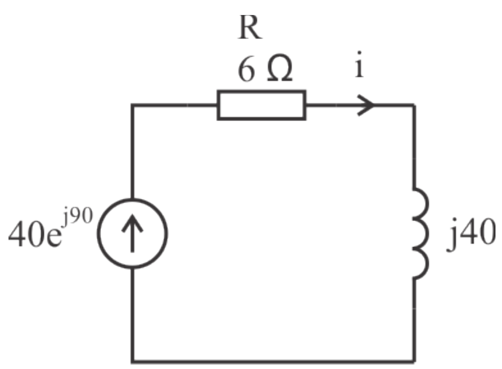
The angular frequency is $\omega=400\,rad/s$, and the reactance is:

$$X_L=\omega L=400.0{,}1=40[\Omega]$$

The phasor of the voltage source is:

$$e^{(2)}(t)=40\sin(400t+90°)\rightarrow \dot{E}^{(2)}=40\,e^{j90°}$$

and the equivalent circuit with complex numbers is:



The complex current is:

$$\dot{I}=\frac{40e^{j90°}}{6+j40}=\frac{40e^{j90°}}{\sqrt{6^2+40^2}e^{j\,atan\frac{40}{6}}}=\frac{40e^{j90°}}{40{,}45e^{j82°}}=0{,}99e^{j8°}[A]$$

and the corresponding sinusoid is:

$$\dot{I}=0{,}99e^{j8°}\rightarrow i^{(2)}(t)=0{,}99\sin(\omega t+8°)[A]$$

Solution b) The total current in the circuit is estimated according to the superposition theorem:

$$i(t)=I^{(0)}+i^{(1)}(t)+i^{(2)}(t)=10+4{,}79\sin(\omega t-73°)+0{,}99\sin(\omega t+8°)[A]$$

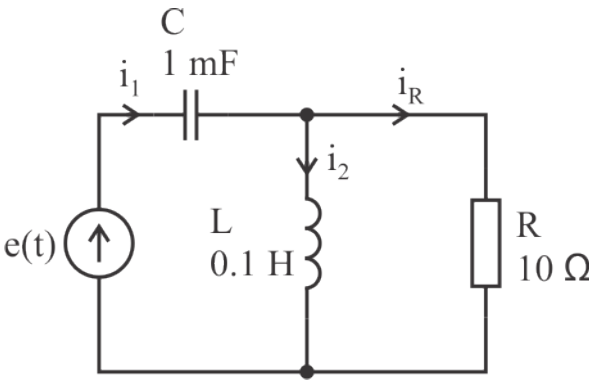
Then the RMS value is:

$$I=\sqrt{(I^{(0)})^2+\frac{(I_m^{(1)})^2+(I_m^{(2)})^2}{2}}=\sqrt{(10)^2+\frac{(4{,}79)^2+(0{,}99)^2}{2}}=10{,}58[A]$$

The active power, dissipated by the resistor, is:

$$P_R=P^{(0)}+P^{(1)}+P^{(2)}=(I^{(0)})R+\left(\frac{I_m^{(1)}}{\sqrt{2}}\right)^2R+\left(\frac{I_m^{(2)}}{\sqrt{2}}\right)^2R=10^2.6+\left(\frac{4{,}79}{\sqrt{2}}\right)^2 6+\left(\frac{0{,}99}{\sqrt{2}}\right)^2 6=672[W]$$

Problem 5. The circuit is powered by nonsinusoidal voltage source $e(t)=10+30\sin(100t)+5\sin(300t)$



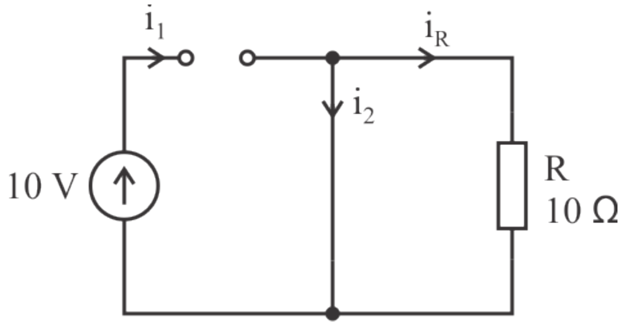
a) For each of the harmonic components obtain the reactances, create an equivalent circuit with complex numbers and estimate the current i_R ;

b) Obtain the instantaneous and RMS current of the resistor $i_R(t)=?\,I_R=?$ and the power dissipated by the resistor $P_R=?$

Solution a)

Only $E^{(0)}=10[V]$ is acting

For DC circuits the capacitors are open circuit and the inductors are short circuit, so the equivalent circuit is:



This means that no current will flow through the resistor:

$$I_R^{(0)}=0[A]$$

<u>Only</u> $e^{(1)}(t)=30\sin(100t)[V]$ <u>is acting</u>

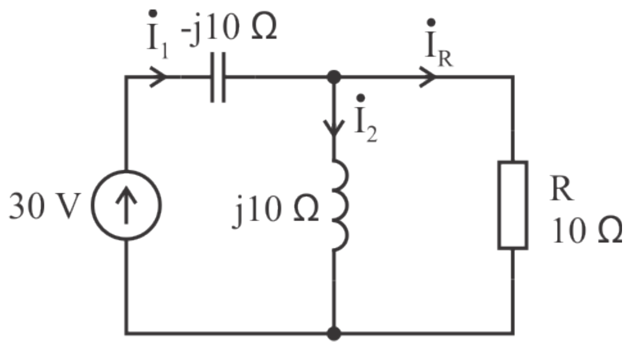
The angular frequency is $\omega=100\,rad/s$. Then the reactances are:

$$X_L=\omega L=100.0{,}1=10[\Omega]$$

$$X_C=\frac{1}{\omega L}=\frac{1}{100.1.10^{-3}}=10[\Omega]$$

and the source phasor is:

$$\dot{E}^{(1)}=30e^{j0}=30[V]$$



We solve the equivalent circuit using Kirchoff's laws:

$$\left|\begin{aligned}\dot{I}_1&=\dot{I}_2+\dot{I}_R\\30&=-j10\dot{I}_1+j10\dot{I}_2\\10\dot{I}_R&-j10\dot{I}_2=0\end{aligned}\right.$$

In matrix form:

$$\begin{vmatrix}\dot{I}_1\\\dot{I}_2\\\dot{I}_R\end{vmatrix}\begin{vmatrix}-1 & 1 & 1\\-j10 & j10 & 0\\0 & -j10 & 10\end{vmatrix}=\begin{vmatrix}0\\30\\0\end{vmatrix}$$

The determinants are:

$$\Delta=\begin{vmatrix}-1 & 1 & 1\\ -j10 & j10 & 0\\ 0 & -j10 & 10\end{vmatrix}=-j100-100+j100=-100$$

$$\Delta_R=\begin{vmatrix}-1 & 1 & 0\\ -j10 & j10 & 30\\ 0 & -j10 & 0\end{vmatrix}=-j300$$

The the current through the resistor is:

$$\dot{I}_R^{(1)}=\frac{\Delta_R}{\Delta}=\frac{-j300}{-100}=j3=3.e^{j90°}[A]$$

and in sinusoidal form:

$$i_R^{(1)}(t)=3\sin(100t+90°)[A]$$

<u>Only $e^{(2)}(t)=5\sin(300t)[V]$ is acting</u>

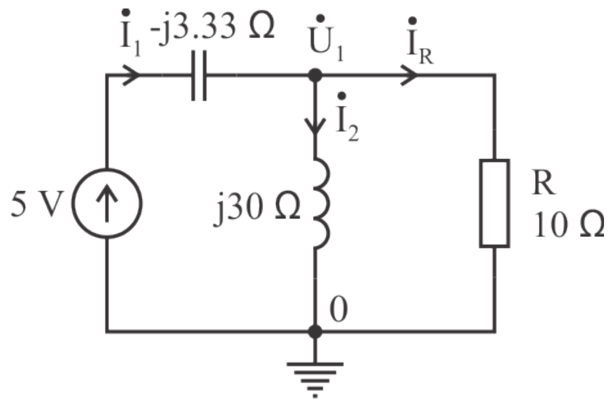
The angular frequency is $\omega=300[rad/s]$ and the reactances are:

$$X_L=\omega L=300*0{,}1=30[\Omega]$$

$$X_C=\frac{1}{\omega L}=\frac{1}{300*1*10^{-3}}=\frac{10}{3}=3{,}33[\Omega]$$

The phasor of the sinusoidal source is:

$$\dot{E}^{(2)}=5e^{j0}=5[V]$$



This time we'll solve the circuit using nodal analysis. There are two nodes, so we have one unknown. We write 1 KCL equation:

$$\dot{I}_1=\dot{I}_2+\dot{I}_R$$

According to Ohm's law the currents are:

$$\dot{I}_1=\frac{0-\dot{U}_1+5}{-j3{,}33}=\frac{5-\dot{U}_1}{-j3{,}33}=j1{,}5-j0{,}3\dot{U}_1$$

$$\dot{I}_2=\frac{\dot{U}_1-0}{j30}=-j0{,}033\dot{U}_1$$

$$\dot{I}_R = \frac{\dot{U}_1 - 0}{10} = 0{,}1\dot{U}_1$$

so our KCL equation becomes:

$$\dot{I}_1 = \dot{I}_2 + \dot{I}_R \rightarrow j1{,}5 - j0{,}3\dot{U}_1 = -j0{,}033\dot{U}_1 + 0{,}1\dot{U}_1$$

The the node voltage is:

$$\dot{U}_1 = \frac{j1{,}5}{0{,}1 + j0{,}267} = \frac{1{,}5e^{j90}}{0{,}285e^{j70}} = 5{,}26e^{j20}$$

an the resistor current is:

$$\dot{I}_R = 0{,}1\dot{U}_1 = 0{,}526e^{j20}$$

In sinusoidal form:

$$\dot{I}_R^{(2)} = 0{,}526e^{j20} \rightarrow i_R^{(2)}(t) = 0{,}526\sin(300t + 20°)[A]$$

Solution b) The instantaneous current through the resistor is:

$$i_R(t) = I_R^{(0)} + i_R^{(1)}(t) + i_R^{(2)}(t) = 0 + 3\sin(100t + 90°) + 0{,}526\sin(300t + 20°)[A]$$

and its RMS value is:

$$I_R = \sqrt{(0)^2 + \frac{(3)^2 + (0{,}526)^2}{2}} = 2{,}154[A]$$

Finally the power dissipated by the resistor is:

$$P_R = P^{(0)} + P^{(1)} + P^{(2)} = \left(I^{(0)}\right)^2 R + \left(\frac{I_m^{(1)}}{\sqrt{2}}\right)^2 R + \left(\frac{I_m^{(2)}}{\sqrt{2}}\right)^2 = 0 + \left(\frac{3}{\sqrt{2}}\right)^2 .10 + \left(\frac{0{,}526}{\sqrt{2}}\right)^2 .10 = 46{,}4[W]$$

It could also be estimated using the RMS current:

$$P_R = I_R^{2.R} = 2{,}154^2 .10 = 46{,}4$$