SINUSOIDAL STEADY STATE CIRCUIT ANALYSIS

1. Introduction

A sinusoidal current has the following form:

$$i(t)=I_m.\sin(\omega t+\varphi)$$

where I_m is the amplitude value;

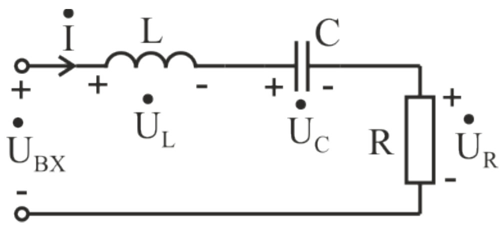
$\omega=2\pi f$ is the angular frequency;

φ is the phase shift.

Each sine could be represented with a phaser and a complex number:

$$i(t)=I_m.\sin(\omega t+\varphi) \quad \rightarrow \quad \dot{I}=I_m.e^{j\varphi}=I_m\cos(\varphi)+jI_m\sin(\varphi)$$

Let's consider a series RLC circuit:



The KVL for the circuit is:

$$\dot{U}_{\text{IN}}=\dot{U}_R+\dot{U}_L+\dot{U}_C$$

where the voltage drops on the resistor, inductor and capacitor are:

$$\dot{U}_R=R.\dot{I}$$

$$\dot{U}_L=jX_L\dot{I}$$

$$\dot{U}_C=-jX_C\dot{I}$$

X_L and X_C are the reactances of the inductor and the capacitor respectively:

$$X_L=\omega L \qquad X_C=\frac{1}{\omega C}$$

Substituting the above equations in the KVL we obtain:

$$\dot{U}_{\text{IN}}=\dot{U}_R+\dot{U}_L+\dot{U}_C=\dot{I}\left(R+j(X_L-X_C)\right)=\dot{I}(R+jX)$$

where $Z=R+jX$ is the complex impedance of the circuit.

$X=X_L-X_C=\omega L-\dfrac{1}{\omega C}$ is the total reactance of the branche.

Every complex number could be presented in rectangular or exponential/polar form, so the complex impedance can also be written as:

$$Z=R+jX=z.e^{j\varphi}=z\angle\varphi$$

where z is the magnitude of the impedance:

$$z=\frac{U_{IN}}{I}=\sqrt{R^2+X^2}$$

φ is the phase difference between the voltage and current in the branch:

$$\varphi=atan\frac{X}{R}$$

2. Problems

Problem 1. Transform the sinusoids in complex form:

- $i_1(t)=10\sin(\omega t+45°)[A]$
- $i_2(t)=12\sin(\omega t-90°)[A]$
- $i_3(t)=3\sin(\omega t+80°)[A]$
- $u_1(t)=100\sin(\omega t)[V]$
- $u_2(t)=50\sin(\omega t-35°)[V]$

Solutions:

$$\dot{I}_1=10e^{j45°}[A]$$

$$\dot{I}_2=12e^{-j90°}[A]$$

$$\dot{I}_3=3e^{j80°}[A]$$

$$\dot{U}_1=100e^{j0°}=100[V]$$

$$\dot{U}_2=50e^{-j35°}[V]$$

Problem 2. Transform the phasors in sinusoidal form:

- $\dot{I}_1=2.25e^{j30°}[A]$
- $\dot{I}_2=1.5e^{-j45°}[A]$
- $\dot{U}_1=5+j10[V]$
- $\dot{U}_2=-10-j10[V]$

Solutions:

$$i_1(t)=2.25\sin(\omega t+30°)[A]$$

$$i_2(t)=1.5\sin(\omega t-45°)[A]$$

$$\dot{U}_1=5+j10=\sqrt{5^2+10^2}e^{j\,arctg\frac{10}{5}}=11,2e^{j63°}\rightarrow u_1(t)=11,2\sin(\omega t+63°)[V]$$

$$\dot{U}_2=-10-j10=-(10+j10)=-\sqrt{(10)^2+(10)^2}e^{j\,arctg\frac{10}{10}}=-14,1e^{j45°}$$

$$\rightarrow u_2(t)=14,1\sin(\omega t+45°)[V]$$

Problem 3. Obtain the reactances of the inductors and capacitors:

- $L_1 = 10\, mH$ at frequency $f = 50\, Hz$
- $C_1 = 0{,}5\, F$ at frequency $f = 110\, Hz$
- $L_2 = 150\, \mu H$ at frequency $f = 10\, kHz$
- $L_3 = 150\, \mu H$ at frequency $f = 100\, kHz$
- $C_2 = 10\, \mu F$ at frequency $f = 10\, kHz$
- $C_3 = 10\, \mu F$ at frequency $f = 100\, kHz$

Solutions:

$$X_{L1} = \omega L_1 = 2\pi f L_1 = 2\pi .50.10.10^{-3} = 3{,}14 [\Omega]$$

$$X_{C1} = \frac{1}{\omega C_1} = \frac{1}{2\pi f C_1} = \frac{1}{2\pi .110.0{,}5} = 0{,}0029 [\Omega]$$

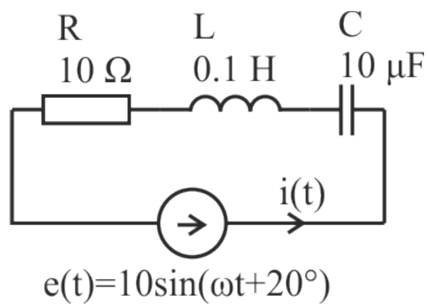
$$X_{L2} = \omega L_2 = 2\pi f L_2 = 2\pi .10000.150.10^{-6} = 9{,}42 [\Omega]$$

$$X_{L3} = \omega L_3 = 2\pi f L_3 = 2\pi .100000.150.10^{-6} = 94{,}2 [\Omega]$$

$$X_{C2} = \frac{1}{\omega C_2} = \frac{1}{2\pi f C_2} = \frac{1}{2\pi .10000.10.10^{-6}} = 1{,}6 [\Omega]$$

$$X_{C3} = \frac{1}{\omega C_3} = \frac{1}{2\pi f C_3} = \frac{1}{2\pi .100000.10.10^{-6}} = 0{,}16 [\Omega]$$

Problem 4. Obtain the current $i(t)$ if the frequency of the source is $f = 100\, Hz$.



First we obtain the reactances of the reactive elements:

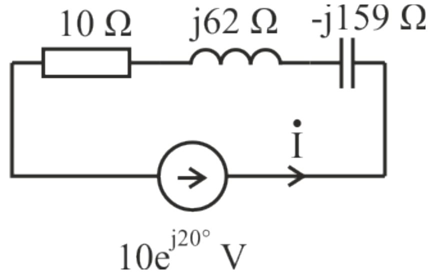
$$X_L = \omega L = 2\pi f L = 2.\pi .100.0{,}1 = 62 [\Omega]$$

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C} = \frac{1}{2.\pi .100.10.10^{-6}} = 159 [\Omega]$$

Next we present the sinusoidal source as phasor:

$$\dot{E} = 10\, e^{j20^\circ} [V]$$

Now we can draw an equivalent circuit with complex numbers:



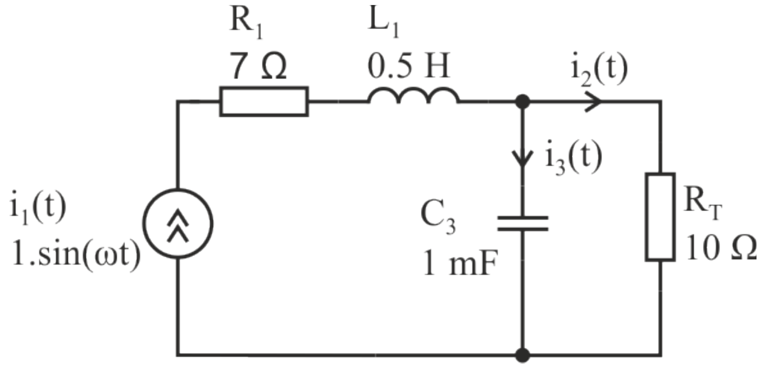
We write the KVL equation for the loop:

$$10e^{j20^\circ}=\dot{I}(10+j60-j159) \quad \rightarrow \quad \dot{I}=\frac{10e^{j20^\circ}}{97{,}5e^{-j84^\circ}}=0{,}103e^{j104^\circ}[A]$$

Finally we transform the phasor in sinusoidal form:

$$\dot{I}=0{,}103e^{j104^\circ} \quad \rightarrow \quad i(t)=0{,}103\sin(\omega t+104^\circ)[A]$$

Problem 5. Obtain the complex currents in the circuit if the frequency of the source is $f=20\,Hz$ and obtain the active power, dissipated by the load R_T .



First we obtain the reactances:

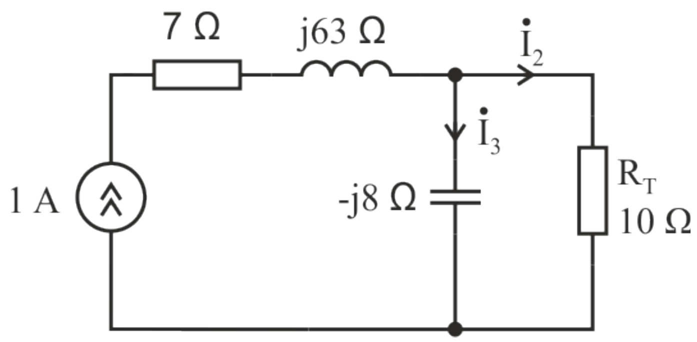
$$X_{L1}=\omega L=2\pi fL=2\pi.20.0{,}5=63[\Omega]$$

$$X_{C3}=\frac{1}{\omega C}=\frac{1}{2\pi fC}=\frac{1}{2\pi.20.1.10^{-3}}=8[\Omega]$$

and present the sinusoidal source as phasor:

$$i_1(t)=1.\sin(\omega t+0) \quad \rightarrow \quad \dot{I}_1=1e^{j0}=1[A]$$

Next we create an equivalent circuit with complex numbers:



For the circuit we can write two equations using Kirchoff's laws method:

$$\left| \begin{aligned} 1 &= \dot{I}_2 + \dot{I}_3 \\ 0 &= 10\dot{I}_2 + j8\dot{I}_3 \end{aligned} \right. \quad \rightarrow \quad \begin{bmatrix} \dot{I}_2 \\ \dot{I}_3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 10 & j8 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The determinants are:

$$\Delta = \begin{bmatrix} 1 & 1 \\ 10 & j8 \end{bmatrix} = 1.j8 - 10 = -10 + j8$$

$$\Delta_2 = \begin{bmatrix} 1 & 1 \\ 0 & j8 \end{bmatrix} = j8$$

$$\Delta_3 = \begin{bmatrix} 1 & 1 \\ 10 & 0 \end{bmatrix} = -10$$

Then the complex currents are:

$$\dot{I}_2 = \frac{\Delta_2}{\Delta} = \frac{j8}{-10+j8} = \frac{8e^{j90°}}{12{,}8e^{-j39°}} = 0{,}63e^{j129°}[A]$$

$$\dot{I}_3 = \frac{\Delta_3}{\Delta} = \frac{-10}{-10+j8} = \frac{-10}{12{,}8e^{-j39°}} = 0{,}78e^{j39°}[A]$$

The corresponding sinusoidal currents are:

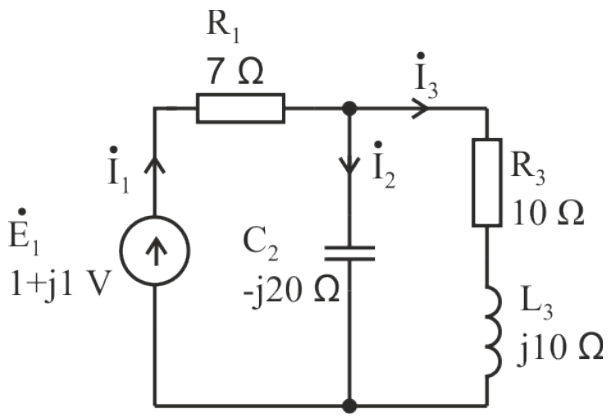
$$\dot{I}_2 = 0{,}63e^{j129°} = \text{¿}\, i_2(t) = 0{,}63\sin(\omega t + 129°)[A]$$

$$\dot{I}_3 = 0{,}78e^{j39°} \quad \rightarrow \quad i_3(t) = 0{,}78\sin(\omega t + 39°)[A]$$

Finally the dissipated power by the load R_T is:

$$P = \left(\frac{I_{2m}}{\sqrt{2}}\right)^2 R_T = \left(\frac{0{,}63}{\sqrt{2}}\right)^2 10 = 1{,}98[W]$$

Problem 6. Obtain the currents in the circuit and the powers of R_3 and L_3.



We'll use nodal analysis. There are two nodes, so we let $\dot{U}_0 = 0$ and need 1 equation:

$$\dot{I}_1 = \dot{I}_2 + \dot{I}_3 \quad \rightarrow \quad \frac{\dot{U}_0 - \dot{U}_1 + 1 + j1}{7} = \frac{\dot{U}_1 - \dot{U}_0}{-j20} + \frac{\dot{U}_1 - \dot{U}_0}{10 + j10}$$

$$0{,}14+j0{,}14-0{,}14\dot{U}_1=j0{,}05\dot{U}_1+0{,}05\dot{U}_1-j0{,}05\dot{U}_1 \quad \rightarrow \quad 0{,}14+j0{,}14=0{,}19\dot{U}_1$$

$$\dot{U}_1=0{,}74+j0{,}74$$

Then our currents are:

$$\dot{I}_1=0{,}14+j0{,}14-0{,}14\dot{U}_1=0{,}05.e^{j45°}\ [A]$$

$$\dot{I}_2=j0{,}05\dot{U}_1=j0{,}05(0{,}74+j0{,}74)=0{,}05.e^{j135°}\ [A]$$

$$\dot{I}_3=(0{,}05-j0{,}05)\dot{U}_1=(0{,}05-j0{,}05)(0{,}74+j0{,}74)=0{,}07\ [A]$$

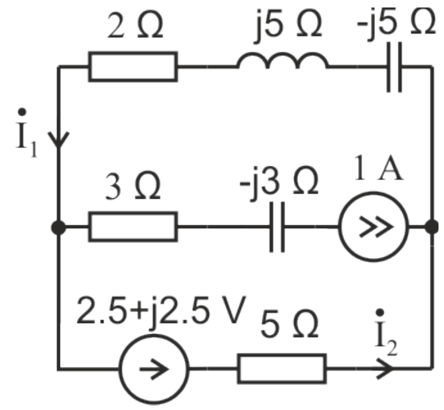
The power dissipated in R_3 is active:

$$P_{R3}=\left(\frac{I_{3m}}{\sqrt{2}}\right)^2.R_3=\left(\frac{0{,}07}{\sqrt{2}}\right)^2.10=0{,}025[W]$$

The power of the inductor is purely reactive:

$$Q_{L3}=\left(\frac{I_{3m}}{\sqrt{2}}\right)^2.X_{L3}=\left(\frac{0{,}07}{\sqrt{2}}\right)^2.10=0{,}025[VAr]$$

Problem 7. Obtain the complex currents in the circuit.



The circuit has 2 unknown currents so we need 2 equations (using Kirchoff's laws):

$$\left|\begin{array}{l} 1+\dot{I}_2=\dot{I}_1 \\ 2{,}5+j2{,}5=5\dot{I}_2+\dot{I}_1(2+j5-5j) \end{array}\right.$$

In matrix form:

$$\begin{bmatrix}\dot{I}_1\\ \dot{I}_2\end{bmatrix}\begin{bmatrix}1 & -1\\ 2 & 5\end{bmatrix}=\begin{bmatrix}1\\ 2{,}5+j2{,}5\end{bmatrix}$$

The determinants are:

$$\Delta=\begin{bmatrix}1 & -1\\ 2 & 5\end{bmatrix}=1.5+1.2=7$$

$$\Delta_1 = \begin{bmatrix} 1 & -1 \\ 2,5 + j2,5 & 5 \end{bmatrix} = 5 + 2.5 + j2.5 = 7,5 + j2,5$$

$$\Delta_2 = \begin{bmatrix} 1 & 1 \\ 2 & 2,5 + j2,5 \end{bmatrix} = 2.5 + j2.5 - 2 = 0,5 + j2,5$$

Then the complex currents are:

$$\dot{I}_1 = \frac{\Delta_1}{\Delta} = \frac{7,5 + j2,5}{7} = 1,07 + j0,36 = 1,13\, e^{j18,6°} [A]$$

$$\dot{I}_2 = \frac{\Delta_2}{\Delta} = \frac{0,5 + j2,5}{7} = 0,07 + j0,36 = 0,37\, e^{j79°} [A]$$