Sinusoidal waveform. Instantaneous and RMS values. Phasors. Resistor, capacitor and inductor in AC circuits. Ohm's and Kirchhoff's laws in AC circuits. Conversation of power in AC circuits. AC circuit analysis.

5.1. Electric waveforms

Circuits with alternating current (AC) are functions whose values vary in both magnitude and direction (fig. 5.1).

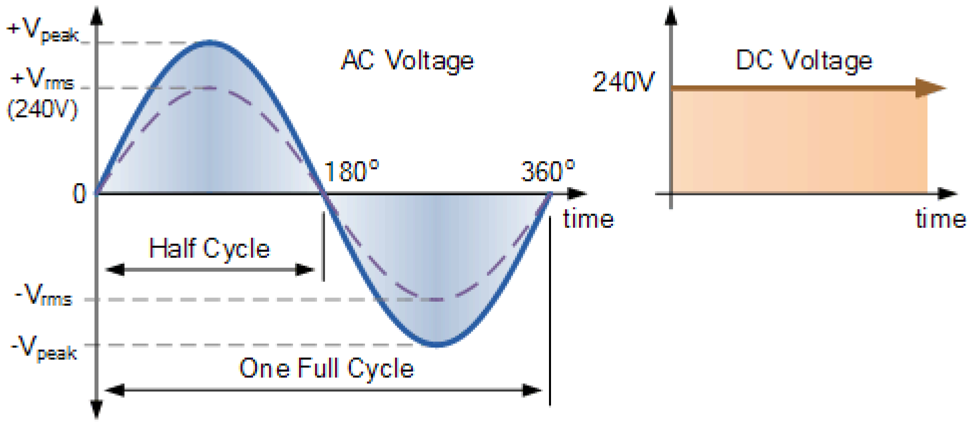


Fig. 5.1.

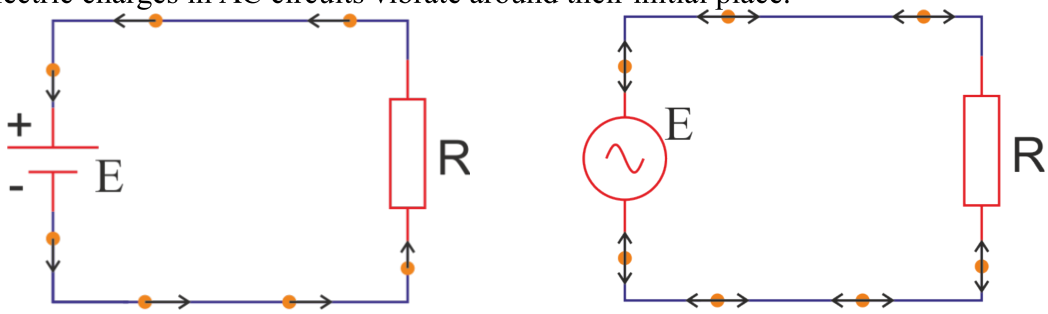
The waveforms are characterized with a couple of quantities:
The period (T) is the length of time that the waveform takes to repeat itself from start to finish. The unit for period is second.
The frequency (f) is the number of times the waveform repeats itself within a one second period:

$$f = \frac{1}{T}$$

The unit for frequency is Hertz (Hz).
The amplitude A is the magnitude or intensity of the signal waveform. The unit of the amplitude depends on the quantity being described: for current it's amps and for voltage – volts.
There are a couple of fundamental differences between AC and DC circuit:

5.1.1 Current flow

- The current flow in DC circuits is always in the same direction;
- In AC circuits the direction of the charge flow changes with time. In other words, the electric charges in AC circuits vibrate around their initial place.



a) b)

Fig. 5.2. Electron flow in DC (a) and AC (b) circuits

5.1.2 Skin effect

- The density of the electric charge flow in DC circuits is the same in the whole volume of a conductor;

- The density of the electric charge flow in AC circuit decreases exponentially in depth, meaning the current density is higher at the conductor surface and lower in depth. This means that the resistance of the conductors for DC and AC are not the same and could be significantly different for very high frequencies. The skin effect is going to be discussed later in the course.

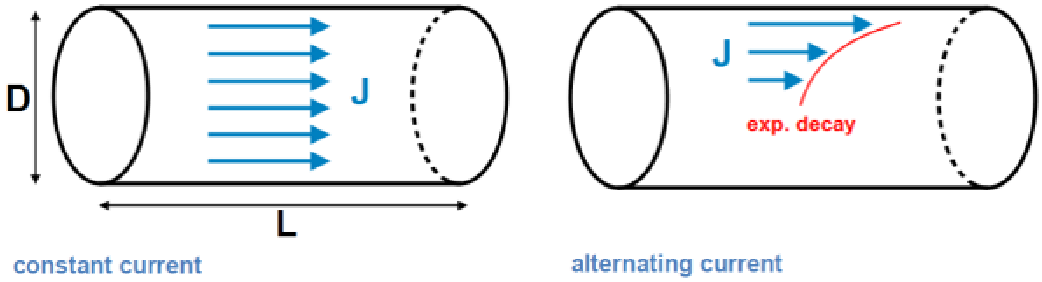


Fig. 5.3. Skin effect in AC circuits.

5.2. Sinusoidal waveform

5.2.1. Basic terms

The most commonly used waveform is the sinusoidal one. Suppose A (t) is a sinusoidal waveform:

$$A(t)=A_m.\sin(\omega t+\varphi)$$

where A_m is the amplitude – it is the minimal and maximal value of the waveform;

$A=\frac{A_m}{\sqrt{2}}$ is the root mean square (RMS), also called effective value;

$\omega=2.\pi.f$ is the angular frequency of the waveform. It is measured in $rad.s^{-1}$;

φ is the phase shift in defrees or radians that the waveform has shifted left or right from the reference point ($t=0$).

When $\varphi=0$ we say that the waveform is in phase. When $\varphi>0$ or $\varphi<0$ the phase is positive or negative respectively (fig. 5.4).

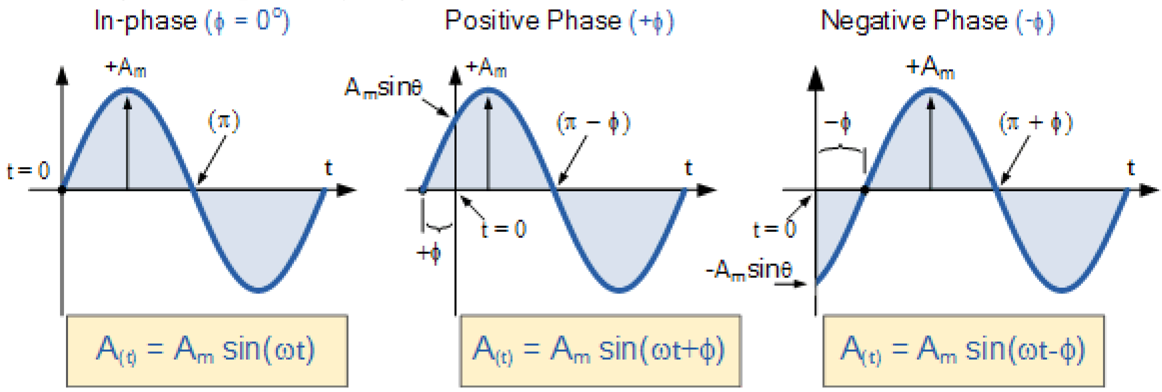


Fig. 5.4.

Consider the current and voltage of a branch are:

$$v(t)=V_m.\sin(\omega t+\varphi_v)$$

$$i(t)=I_m.\sin(\omega t+\varphi_i)$$

where φ_v and φ_i are their phase angles.

The difference $\varphi_v-\varphi_i$ is called phase difference (fig. 5.5):

$$\varphi=\varphi_v-\varphi_i$$

- If $\varphi>0$ then the current lags the voltage by φ ;

- If $\varphi<0$ then the current leads the voltage by φ;
- If $\varphi=0$ then the current and voltage are "in phase".

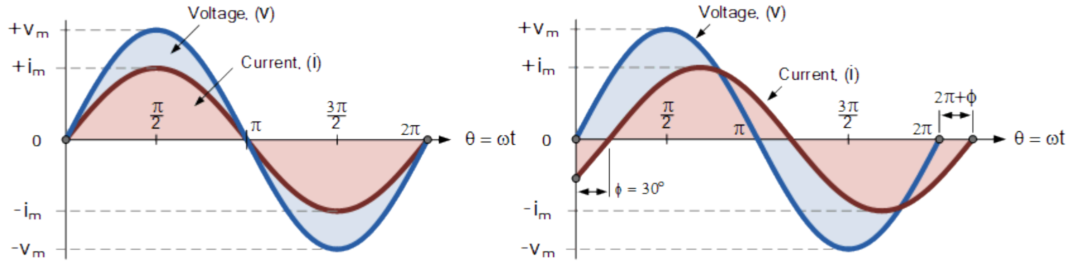


Fig. 5.5.

5.2.2. Sinusoids and phasors

The sinusoidal waveform $A(t)=A_m.\sin(\omega t+\varphi)$ could be expressed as a vector, rotating anti-clockwise with an angular frequency ω (fig. 5.5).

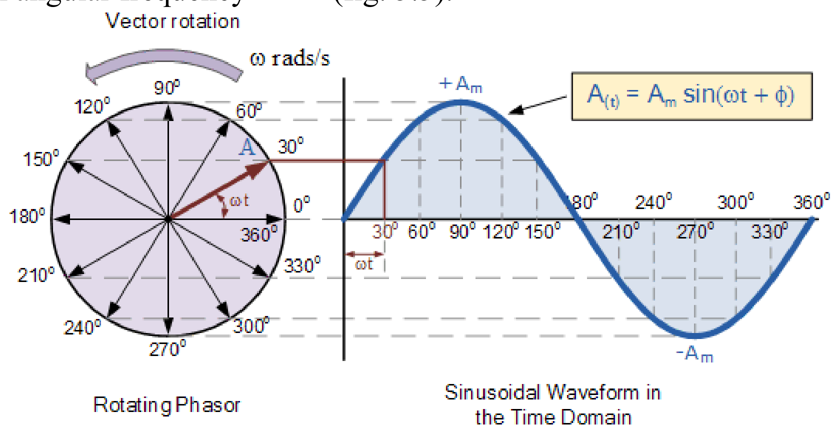


Fig. 5.5.

As can be seen when the time is $t=0$ the vecor is rotated at $0°$, $180°$ and $360°$. Similarly when $A(t)$ has a maximum ($+A_m$) the vector is rotated at $90°$ and when $A(t)$ has a minimum ($-A_m$) – the vector is rotated at $-90°$.

Consider the current and voltage of a branch are:

$$v(t)=V_m.\sin(\omega t)$$

$$i(t)=I_m.\sin(\omega t-30°)$$

The current lags the voltage by $\varphi=30°$ (fig. 5.6a). Then the phasor diagram of the two vectors for $t=0$ is presented in fig. 5.6b. In time the two vectors rotate together with angular frequency ω however the current vector will continue to lag the voltage by $30°$ (fig. 5.6c).

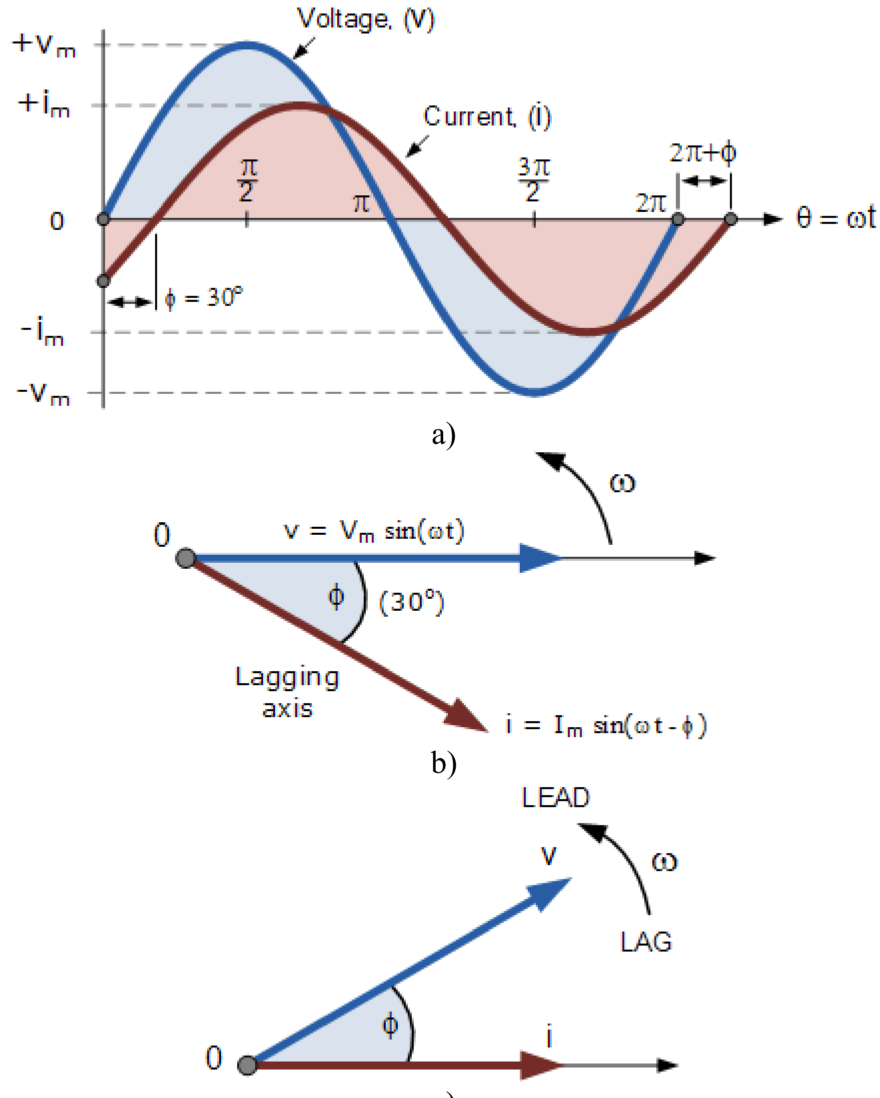


Fig. 5.6.

The sinusoidal waveform $A(t)=A_m.\sin(\omega t+\varphi)$ could be expressed in phasor form as:

$$\dot{A}_m=A_m.e^{j\varphi}=A_m\cos(\varphi)+j\,A_m\sin(\varphi)$$

where $\dot{A}_m$ is also called complex amplitude.

The above equation is called the Euler's formula (fig. 5.7).

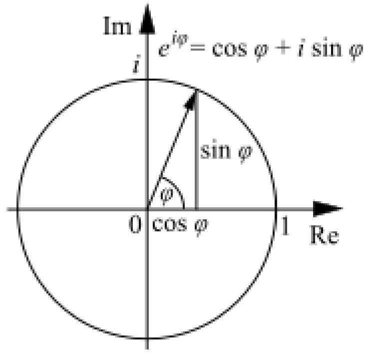


Fig. 5.7.

Example: Obtain phasors of $v_1(t)=6.\sin(\omega t+60°),V$ and $v_2(t)=-6.\sin(\omega t+30°)$.

$$\dot{V}_{m1}=6.e^{j60°}=6\cos(60°)+j6\sin(60°)=3+j5.20,V$$

$$\dot{V}_{m2}=-6.e^{j30°}=-6\cos(30°)-j6\sin(30°)=-5.20-j3$$

We can also use the RMS phasors:

$$\dot{V}_1=\frac{6}{\sqrt{2}}e^{j60°}=4.24\cos(60°)+j4.24\sin(60°)=2.12+j3.67,V$$

$$\dot{V}_2=\frac{-6}{\sqrt{2}}.e^{j30°}=-4.24\cos(30°)-j4.24\sin(30°)=-3.67-j2.12,V$$

Example: Obtain the sinusoids of the peak phasors $\dot{V}_{m1}=2+j5$ and the RMS phasor $\dot{V}_2=5-j1$.

First we convert the phasors in polar form:

$$\dot{V}_{m1}=2+j5=\sqrt{2^2+5^2}.e^{j\,atan\frac{5}{2}}=5.34\,e^{j68.2°}$$

$$\dot{V}_2=5-j1=\sqrt{5^2+1^2}.e^{j\,atan\frac{-1}{5}}=5.1\,e^{-j11.3°}$$

Then we can write down the sine values:

$$v_1(t)=5.34\sin(\omega t+68.2°),V$$

$$v_2(t)=5.1.\sqrt{2}\sin(\omega t-11.3°)=7.21.\sin(\omega t-11.3°),V$$

5.3. Sinusoidal steady state

5.3.1. Passive elements

5.3.1.1 Resistors

Consider an ideal resistor powered by a sine voltage source so that the current $i_R(t)$ of the resistor has a phase shift $\varphi_I=0$:

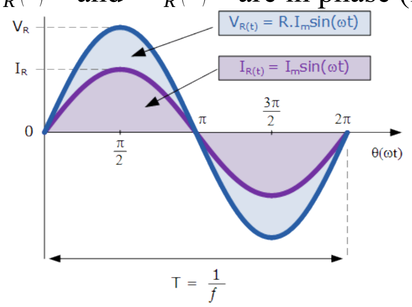
$$i_R(t)=I_m.\sin(\omega t)$$

According to Ohm's law the voltage drop $v_R(t)$ is:

$$v_R(t)=i_R(t).R=I_m.R.\sin(\omega t)$$

The phase difference is $\varphi=0$ which means that $v_R(t)$ and $i_R(t)$ are in phase (fig 4.8).

a)



b)

Fig.4.8.

5.3.1.2 Inductors

Consider an ideal inductor L is powered by a sine voltage source which creates the current $i_L(t)$ with phase shift $\varphi_I=0$:

$$i_L(t)=I_m.\sin(\omega t)$$

The voltage drop on L is:

$$v_L(t)=L.\frac{di_L(t)}{dt}=L.\frac{d(I_m.\sin(\omega t))}{dt}=\omega L.I_m.\cos(\omega t)=\omega L.I_m.\sin(\omega t+90°)$$

The above equations show that $v_L(t)$ leads $i_L(t)$ by $90°$ (fig. 5.9). Another important observation is that the "resistance" of the inductor in sinusoidal steady state is ωL .

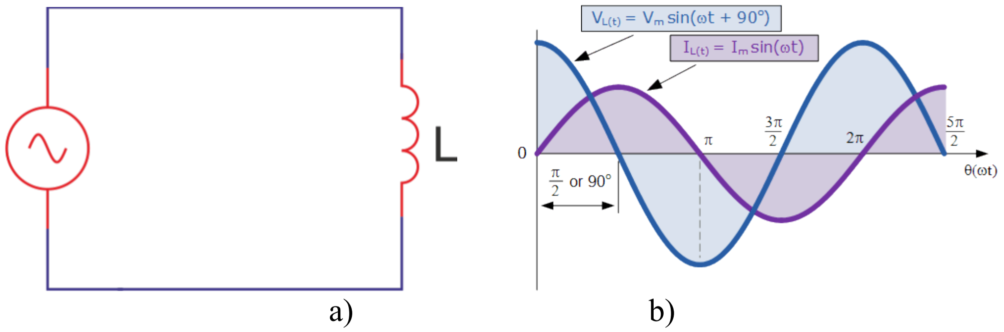


Fig. 5.9.

5.3.1.3 Capacitors

Consider an ideal capacitor C is powered by a sine voltage source which creates the current $i_C(t)$ with phase shift $\varphi_I = 0$:

$$i_C(t) = I_m . \sin(\omega t)$$

The votlage drop on the capacitor is:

$$v_C(t) = \frac{1}{C} \int i_C(t) dt = \frac{1}{C} \int I_m . \sin(\omega t) dt = \frac{-1}{\omega C} I_m . \cos(\omega t) = \frac{-1}{\omega C} I_m . \sin(\omega t + 90°) = \frac{1}{\omega C} I_m . \sin(\omega t - 90)$$

The equations show that the voltage $v_C(t)$ lags the current $i_C(t)$ by 90 . It can also be seen that the "resistance" of the capacitor is $\frac{1}{\omega C}$ (fig. 5.10).

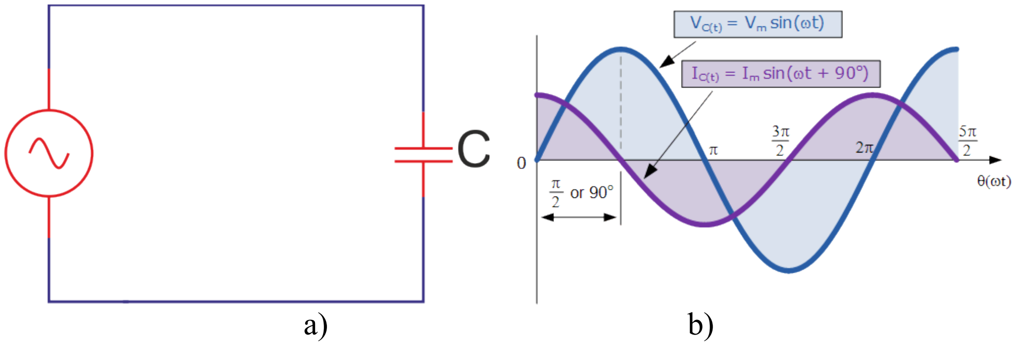


Fig. 5.10.

5.3.1.4 Series RC circuit

Consider a series RC circuit powered by a sine voltage source $v(t)$ (fig. 5.11a), which creates the current $i(t) = I_m . \sin(\omega t)$. The voltage drops on the capacitor and the resistor are:

$$v_R(t) = R . I_m . \sin(\omega t)$$

$$v_C(t) = \frac{1}{\omega C} I_m . \sin(\omega t - 90)$$

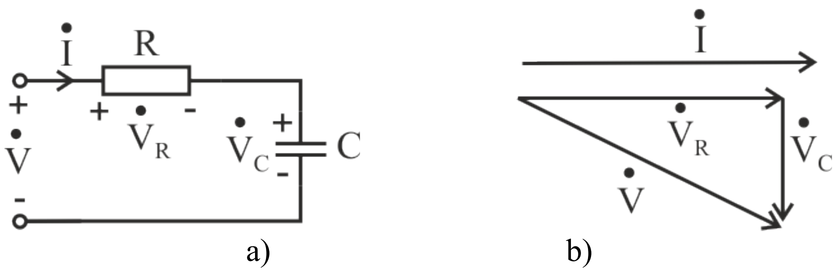


Fig. 5.11

The above equations could be written with phasors as:

$$\dot{I}=\frac{I_m}{\sqrt{2}}e^{j0}=I$$

$$\dot{V}_R=R\,I\,e^{j0}=R\,\dot{I}$$

$$\dot{V}_C=\frac{1}{\omega C}I_m e^{-j90°}=X_C e^{-j90°}\dot{I}$$

The vector diagram of of the voltages and current are presented in fig. 5.11b. It could be seen that the source voltage is:

$$\dot{V}=\dot{V}_R+\dot{V}_C=\dot{I}\left(R+X_C e^{-j90°}\right)=\dot{I}\left(R-j\,X_C\right)$$

In other words the complex impedance of the capacitor is $-j\,X_C$ or $-j\frac{1}{\omega C}$ where $\frac{1}{\omega C}$ is the capacitive reactance of the capacitor in Ohms.

5.3.1.5 Series RL circuit

Consider a series RL circuit powered by a sinusoidal voltage source $v(t)$ (fig. 5.12a), which creates the current $i(t)=I_m.\sin(\omega t)$. The phasor values are:

$$\dot{I}=\frac{I_m}{\sqrt{2}}e^{j0}=I$$

$$\dot{V}_R=R\,I\,e^{j0}=R\,\dot{I}$$

$$\dot{V}_L=\omega L\,I\,e^{j90°}=X_L e^{j90°}\dot{I}$$

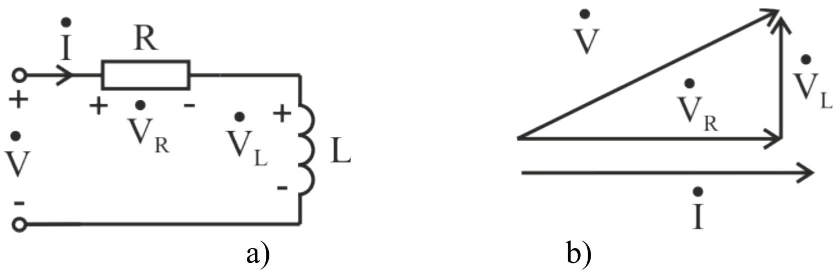


Fig. 5.12

The vector diagram is presented in fig. 5.12b and the source voltage is:

$$\dot{V}=\dot{V}_R+\dot{V}_L=\dot{I}\left(R+X_L e^{j90°}\right)=\dot{I}\left(R+j\,X_L\right)$$

The quantity $j\,X_L=j\omega L$ is the complex impedance of the inductor where ωL is the inductive reactance in Ohms.

5.3.1.6 Series RLC circuit

Consider a series RLC circuit powered by a sine voltage source $v(t)$, which creats a current $i(t)=I_m.\sin(\omega t)$ (4.13). As was already demonstrated the phasor voltage drops on the three passive elements are:

$$\dot{I}=\frac{I_m}{\sqrt{2}}e^{j0}=I$$

$$\dot{V}_R=\dot{I}\,R$$

$$\dot{V}_L=\dot{I}\,\omega L\,e^{j90°}=\dot{I}\,X_L e^{j\,90°}$$

$$\dot{V}_C=\dot{I}\frac{1}{\omega C}e^{-j90°}=\dot{I}\,X_C\,e^{-j90°}$$

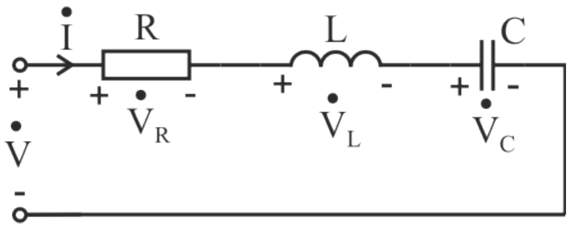


Fig. 5.13.

From the above equations we can write:

$$\dot{V}=\dot{V}_R+\dot{V}_L+\dot{V}_C=\dot{I}\left(R+jX_L-jX_C\right)=\dot{I}\left(R+jX\right)=\dot{I}.Z$$

where $X=X_L-X_C$ is the reactance of the circuit in Ohms.

$Z=R+jX$ is the complex impedance of the circuit in Ohms.

The complex impedance can also be presented as:

$$Z=R+jX=z.e^{j\varphi}$$

where $z=\sqrt{R^2+X^2}$ is the impedance of the circuit in Ohms;

$\varphi=\tan^{-1}\frac{X}{R}$ is the phase difference of the circuit in degrees or radians.

The vector diagram of the RLC circuit could have three forms depending on the values of X_L and X_C (fig. 5.14):

- If $\frac{1}{\omega C}>\omega L$ the reatance X is negative and so is the phase difference: $\varphi<0$ (fig. 5.14a);
- If $\frac{1}{\omega C}=\omega L$ the reactance X is zero and so is the phase difference: $\varphi=0$ (fig. 5.14b).
- If $\frac{1}{\omega C}<\omega L$ the reactance X and the phase difference are positive: $\varphi>0$ (fig. 5.14c);

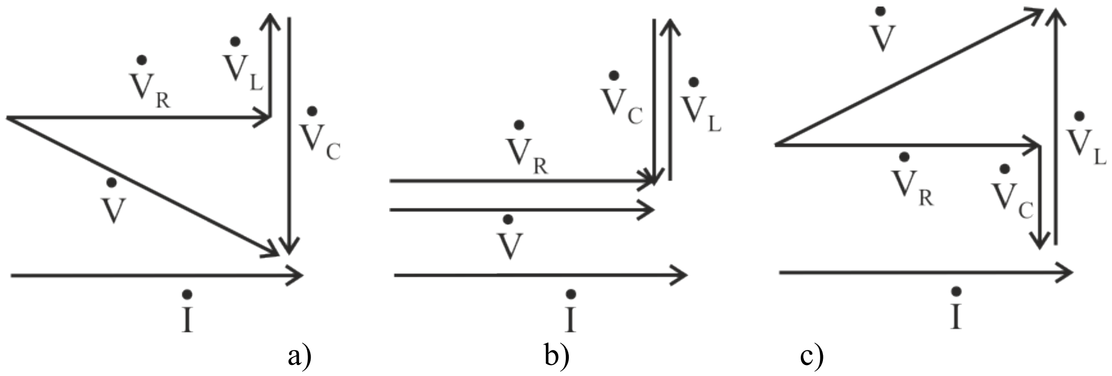


Fig. 5.14. Vector diagrams of a series RLC circuit: a) $\frac{1}{\omega C}>\omega L$; b) $\frac{1}{\omega C}=\omega L$; c) $\frac{1}{\omega C}<\omega L$.

The situation when $\frac{1}{\omega C}=\omega L$ is called series resonance and will be considered more closely later.

From the vector diagrams could be seen that the resistances of a RLC circuit is given by the sides of a right triangle (fig. 5.15).

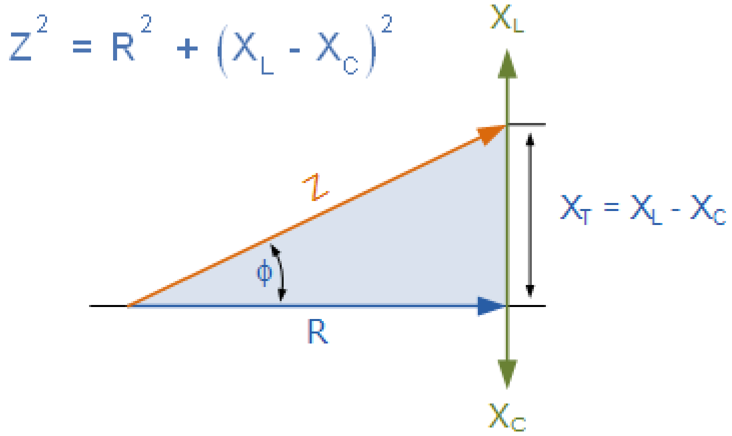


Fig. 5.15.

Example: Estimate the complex impedance of a RLC circuit ($R=1k\Omega$, $L=0.1\,H$, $C=10\,\mu F$) which is powered by a sinusoidal source with frequency $f=50\,Hz$.
The angular frequency is:

$$\omega=2.\pi.f=314\,rad/s$$

Then we can estimate the inductive and capacitive reactance:

$$X_L=314\times 0.1=31.4\,\Omega$$

$$X_c=\frac{1}{314\times 10.10^{-6}}=\frac{10^{-6}}{3140}=318\,\Omega$$

The complex impedance is:

$$Z=1000+j(31.4-318)=1000-j\,286{,}6=1040.e^{-j16^\circ}\,\Omega$$

From the above equation is also seen that the impedance is $z=1040\,\Omega$ and the phase difference is $\varphi=-16^\circ$.

5.3.1.7 Parallel RLC circuit

Consider the parallel RLC circuit presented in fig. 5.16, powered by the voltage source $v(t)=V_m.\sin(\omega t)$. The currents in the circuit are:

$$i_R(t)=\frac{v(t)}{R}=v(t)G=V_m.\sin(\omega t)G$$

$$i_L(t)=\frac{1}{L}\int v(t).dt=\frac{1}{\omega L}V_m.\sin(\omega t-90)$$

$$i_C(t)=C\frac{dv(t)}{dt}=\omega C.V_m\cos(\omega t)$$

where $G=\frac{1}{R}$ is the conductance of the resistor in Siemens;

$\frac{1}{\omega L}$ is the inductive susceptance, which is the reciprocal of the inductive reactance in Siemens;

ωC is the capacitive susceptance in Siemens.

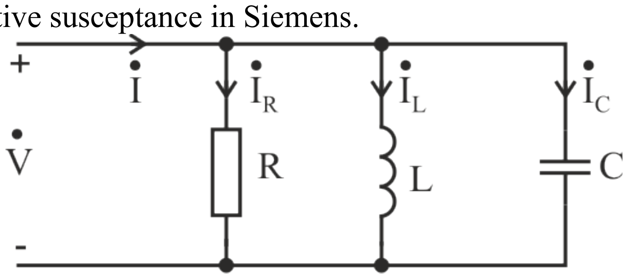


Fig. 5.16.

The above equations could be written with phasors as:

$$\dot{V}=\frac{V_m}{\sqrt{2}}e^{j0}=V$$

$$\dot{I}_R=\dot{V}G$$

$$\dot{I}_L=\dot{V}\frac{1}{\omega L}e^{-j90°}=\dot{V}B_L e^{-j90°}$$

$$\dot{I}_C=\dot{V}\omega C e^{j90°}=\dot{V}B_C e^{j90°}$$

Then the total current coming from the source is:

$$\dot{I}=\dot{I}_R+\dot{I}_L+\dot{I}_C=\dot{V}\left(G-j\left(B_L-B_C\right)\right)=\dot{V}\left(G-jB\right)=\dot{V}.Y$$

where $B=B_L-B_C$ is the susceptance of the circuit in Siemens.

$Y=G-jB$ is the complex admitance of the circuit in Siemens.

The complex admitance can also be presented as:

$$Y=G-jB=y.e^{-j\varphi}$$

where $y=\sqrt{G^2+B^2}$ is the admitance of the circuit.

The vector diagram of the parallel RLC circuit could have three forms depending on the values of B_L and B_C (fig. 5.17):

- If $\omega C<\frac{1}{\omega L}$ then the phase difference is positive $\varphi>0$ (fig. 5.17a);
- If $\omega C=\frac{1}{\omega L}$ then the susceptance B is zero and so is the phase difference: $\varphi=0$ (fig. 5.17b).
- If $\omega C>\frac{1}{\omega L}$ then the phase difference is negative: $\varphi<0$ (fig. 5.17c);

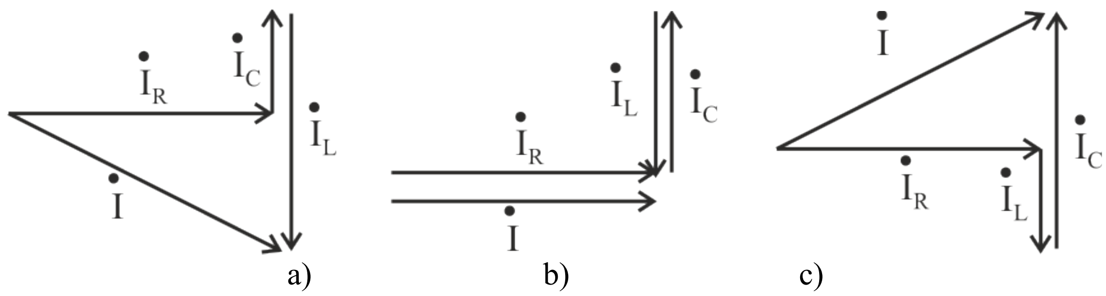


Fig. 5.17. Vector diagrams of a parallel RLC circuit: a) $\omega C<\frac{1}{\omega L}$; b) $\omega C=\frac{1}{\omega L}$; c) $\omega C>\frac{1}{\omega L}$.

The situation when $\omega C=\frac{1}{\omega L}$ is called parallel resonance and will be examined more closely later.

From the vector diagrams it could be seen that the conductance, susceptance and admitance are connected by the sides of a right triangle (fig. 5.18).

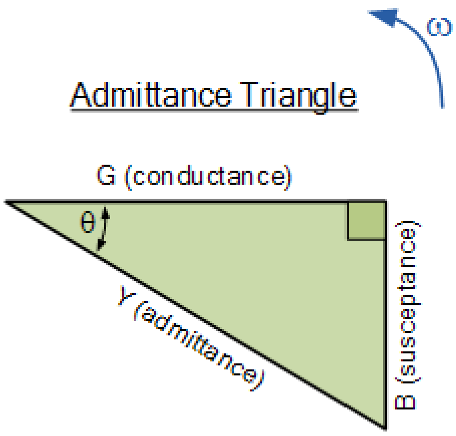


Fig. 5.18.

5.3.2. Ohm's and Kirchhoff's laws

The main circuit laws in sinusoidal steady state have already been applied in the above analysis and now they will be described in details.

Ohm's law

Ohm's law for sinusoidal circuits can be written for the RMS current I and voltage V and the impedance of the circuit:

$$V=z.I$$

where the impedance is $z=\sqrt{R^2+X^2}$

In complex form Ohm's law is:

$$\dot{V}=Z.\dot{I}$$

where $Z=R+j\left(X_L-X_C\right)=R+j\left(\omega L-\frac{1}{\omega C}\right)$ is the complex impedance of the branch.

Kirchhoff's current law

Consider the node presented in fig. 5.19. The instantaneous values of the currents are related according to Kirchhoff's current law:

$$i_1(t)=i_2(t)+i_3(t)$$

Note that the above equation means that in every moment of time the total current/charge flow through the node is 0. The above equation could be written with phasors as:

$$\dot{I}_1=\dot{I}_2+\dot{I}_3$$

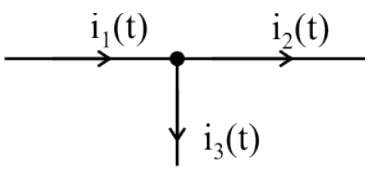


Fig. 5.19.

In general form the KCL for any node in complex form is:

$$\sum \dot{I}_{IN}=\sum \dot{I}_{OUT}$$

Kirchhoff's voltage law

Consider a series RLC circuit powered by a voltage source $v(t)$ with instantaneous values of the current and the voltage drops (fig. 5.20):

$$i(t)=I_m.\sin(\omega t)$$

$$v_R(t)=R.I_m.\sin(\omega t)$$
$$v_L(t)=L.\frac{di(t)}{dt}=\omega L.I_m.\sin(\omega t+90°)=\omega L.I_m\cos(\omega t)$$
$$v_C(t)=\frac{1}{C}.\int i(t).dt+u_C(0)=\frac{1}{\omega C}I_m.\sin(\omega t-90)=\frac{-1}{\omega C}I_m\cos(\omega t)$$

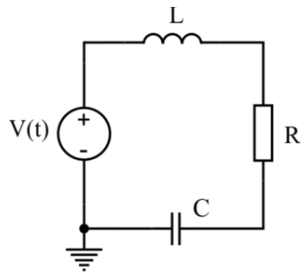


Fig. 5.20.

The KVL for the loop is:

$$v(t)=R.I_m.\sin(\omega t)+L.\frac{di(t)}{dt}+\frac{1}{C}.\int i(t).dt+u_C(0)=I_m\left(\sin(\omega t)R+\cos(\omega t)\left(\omega L-\frac{1}{\omega C}\right)\right)$$

The above equation could be written in complex form as:

$$\dot{V}=\dot{V}_R+\dot{V}_L+\dot{V}_C=\dot{I}\left(R+j\left(\omega L-\frac{1}{\omega C}\right)\right)=\dot{I}.Z$$

In the general case KVL could be written for any closed loop as:

$$\sum \dot{V}_{SRC_n}=\sum \dot{V}_k=\sum \dot{I}_k.Z_k$$

where $\dot{V}_k$, $\dot{I}_k$ and Z_k are the voltage drop, the current and the complex resistance of the k[th] branch and $\dot{V}_{SRC_n}$ is the n[th] voltage source.

5.3.3. Power in sinusoidal steady state circuits

5.3.3.1. Instantaneous and average power

Instantaneous power in a resistive circuit

Consider a purely resistive circuit powered by a sine source where the current and voltage drop of the resistor are in phase (fig. 5.21):

$$i(t)=I_m.\sin(\omega t)$$
$$v(t)=V_m.R.\sin(\omega t)$$

Then the instantaneous power is:

$$p(t)=i(t).v(t)=I_mV_m\sin^2(\omega t)$$

Considering $\sin^2(\omega t)$ is always positive so is $p(t)$, which can also be seen from the time diagram. In other words the instantaneaous consumed power is always positive.

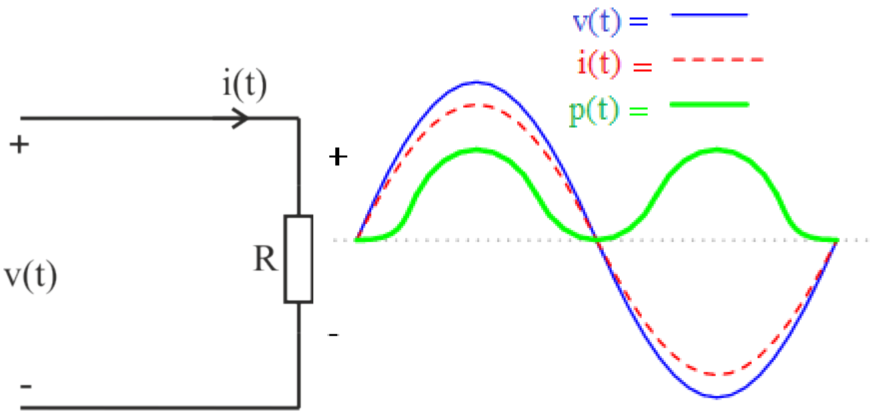


Fig. 5.21.

The average power consumption of a resistor can be obtained through integration of $p(t)$ for one waveform period:

$$P_{AVG}=\frac{1}{T}\int_{0}^{T} p(t)\,dt=\frac{1}{T}\int_{0}^{T} I_m V_m \sin^2(\omega t)\,dt=\frac{I_m V_m}{2}$$

P_{AVG} is non zero and this kind of power is called active or real power.

Instantaneous power in pure capacitive circuits

Consider the purely capacitive circuit in fig. 5.22 where the current and voltage are:

$$i(t)=I_m \sin(\omega t)$$
$$v(t)=V_m \sin(\omega t-90°)$$

Then the instantaneous power is:

$$p(t)=i(t).v(t)=I_m V_m \frac{-\cos(2\omega t-90°)}{2}=\frac{-I_m V_m}{2}\sin(2\omega t)$$

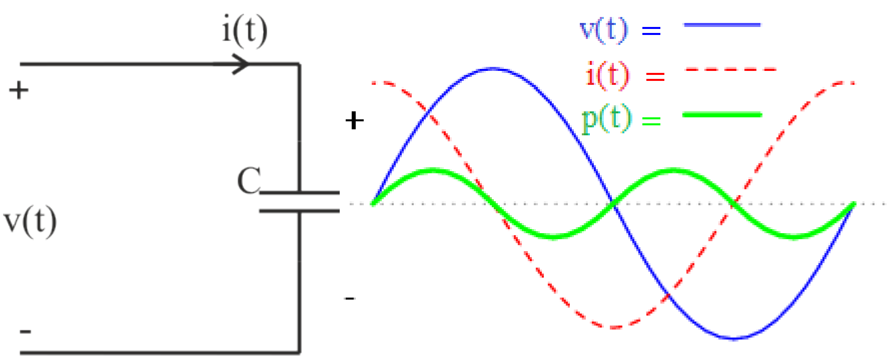


Fig. 5.22

The average power consumption of the capacitor can be obtained through integration over one period T :

$$P_{AVG}=\frac{1}{T}\int_{0}^{T} p(t)\,dt=\frac{1}{T}\int_{0}^{T} \frac{-I_m V_m}{2}\sin(2\omega t)\,dt=0$$

P_{AVG} is zero which means that the ideal capacitor does not consume power but simply stores it as an electric field and later returns it back to the source. Such kind of power is called reactive power.

Instantaneous power in pure inductive curcuits

Consider the purely inductive circuit in fig. 5.23 where the current and voltage are:

$$i(t)=I_m.\sin(\omega t)$$
$$v(t)=V_m.\sin(\omega t+90°)$$

Then the instantaneous power is:

$$p(t)=i(t).v(t)=I_mV_m\frac{-\cos(2\omega t+90°)}{2}=\frac{I_mV_m}{2}\sin(2\omega t)$$

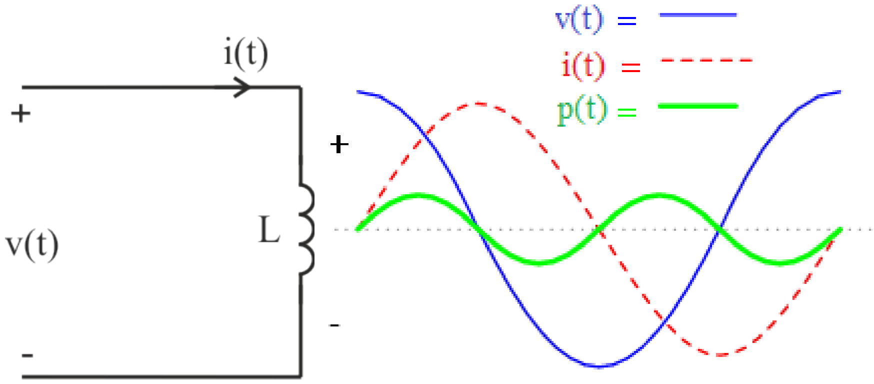


Fig. 5.23

The average power consumption of the inductor can be obtained through integration over one period T :

$$P_{AVG}=\frac{1}{T}\int_0^T p(t)\,dt=\frac{1}{T}\int_0^T I_mV_m\sin(2\omega t)\,dt=0$$

P_{AVG} is zero meaning that the ideal inductor does not consume power. It stores it as magnetic energy and later returns it back. Once again this kind of energy is called reactive.

Instantaneaous power in series RLC circuit

Consider the series RLC circuit in fig. 5.24 where the current and voltage are:

$$v(t)=V_m.\sin(\omega t)$$
$$i(t)=I_m.\sin(\omega t-\varphi)$$

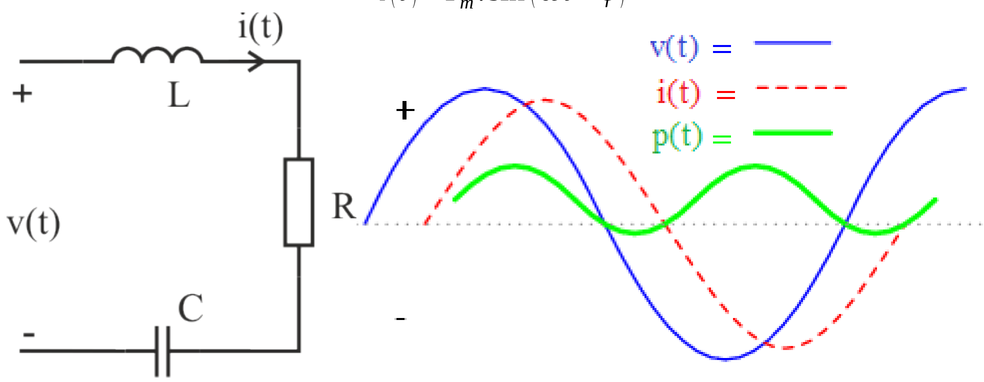


Fig. 5.24.

The instantaneous power is:

$$p(t)=i(t).v(t)=V_mI_m\cos(\varphi)\sin^2(\omega t)-V_mI_m\sin(\varphi)\sin(\omega t)\cos(\omega t)$$

The average power can be obtained through integration fir one cycle of the sinusoidal function:

$$P_{AVG}=\frac{1}{T}\int_0^T p(t)\,dt=\frac{V_m}{\sqrt{2}}\frac{I_m}{\sqrt{2}}\cos(\varphi)=VI\cos(\varphi)$$

It can be seen that the average power has a maximum when $\varphi=0°$ ($X=0$) and a minimum when $\varphi=90°$ ($R=0$).

The values $V=\frac{V_m}{\sqrt{2}}$ and $I=\frac{I_m}{\sqrt{2}}$ are the effective (also called root mean square or RMS) values of the voltage and current. They can also be estimated from the root mean square of the halfcycle of a periodic wave (fig. 5.25):

$$V=\sqrt{\frac{V_1^2+V_2^2+\ldots+V_{12}^2}{12}}$$

For a sinusoidal waveform the above equation is:

$$V=\frac{V_m}{\sqrt{2}}$$

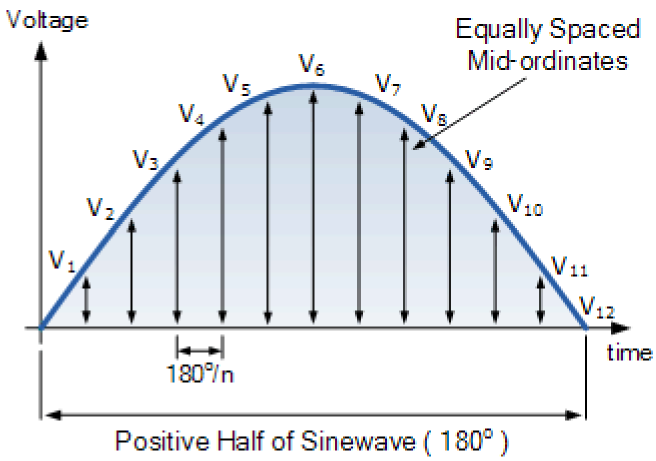


Fig. 5.25.

5.3.3.2. Active, reactive, apparent and complex power

As was earlier demonstrated power consumed in resistors is really consumed and called active power, while power consumed in reactive elements is only temporary stored and it's called reactive. Similarly to the resistance trangle the powers in a AC circuit are related by the sindes of a right trianle (fig. 5.26).

The active power, measured in Watts [W] is:

$$P=V.I.\cos\varphi$$

The reactive power, measured in VArs is:

$$Q=V.I.\sin\varphi$$

S is called the apparent power and is measured in VA:

$$S=V.I$$

The apparent power and the phase difference angle could be estiamted from the right triangle as well:

$$S=\sqrt{P^2+Q^2}$$

$$\varphi=\tan^{-1}\frac{Q}{P}$$

The phase different angle could be positive or negative and so can the reactive power while the active power is always positive.

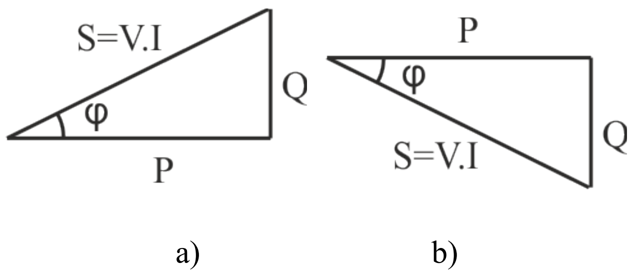


Fig. 5.26. Power triangle for: a) inductive load; b) capacitive load.

The phasor describing the power triangle is called complex power and is equal to:

$$\dot{S}=P+jQ=S.e^{j\varphi}$$

Similarly to the apparent power the complex power is measured in VA.

5.3.3.3. Conservatino of power in AC circuits

Following the conservation of energy law power should be conserved in AC circuits. This conservation is defined as:

$$\sum \dot{S}_{SRC} = \sum \dot{S}_{CONS}$$

where $\dot{S}_{SRC}$ and $\dot{S}_{CONS}$ are the cumulative complex powers of the sources and consumers respectively. Since $\dot{S} = P + jQ$, the above equation could also be written individually for the active and reactive power in the circuit:

$$\sum P_{SRC} = \sum P_{CONS}$$

and

$$\sum Q_{SRC} = \sum Q_{CONS}$$

Power of the consumers

The active and reactive power for a certain branch of the circuit can be estimated respectively with:

$$P = I^2 . R Q = I^2 . \left(\omega L - \frac{1}{\omega C} \right)$$

Power of the sources

Consider the current flow through a voltage source $\dot{V}_V$ is $\dot{I}_V$. Then the complex power of the voltage source is:

$$\dot{S}_V = \dot{V}_V . \overset{*}{I}_I$$

where $\overset{*}{I}_I$ is the complex conjugate of the current:

$$\dot{I}_V = I_V . e^{j\varphi} \rightarrow \overset{*}{I}_I = I_V . e^{-j\varphi}$$

Next consider the current source $\dot{I}_I$ with a voltage drop $\dot{U}_I$. The complex power of the votlage is:

$$\dot{S}_I = \dot{V}_I . \overset{*}{I}_I$$

5.3.3.4. Maximum active power transfer theorem for AC circuits

Consider a load $Z_{LOAD} = R_{LOAD} + j X_{LOAD}$ is powered through a AC circuit, whose Thevenin equivallent is presented in fig. 5.27 where the Thevenin equivallent impedance is $Z_{Th} = R_{Th} + j X_{Th}$.

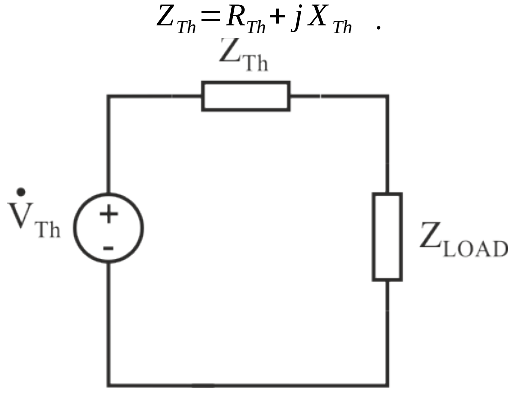


Fig. 5.27. Thevenin equivallent circuit powering a load.

The complex current in the circuit is:

$$\dot{I} = \frac{\dot{V}_{Th}}{Z_{LOAD} + Z_{Th}} = \frac{\dot{V}_{Th}}{(R_{LOAD} + R_{Th}) + j(X_{LOAD} + X_{Th})} = \frac{\dot{V}_{Th}}{Z_{Eq}}$$

The equivallent resistance Z_{Eq} can be presented in polar form as:

$$Z_{Eq}=\sqrt{\left(R_{LOAD}+R_{Th}\right)^2+\left(X_{LOAD}+X_{Th}\right)^2}e^{j\,atan\frac{X_{LOAD}+X_{Th}}{R_{LOAD}+R_{Th}}}=z.e^{j\varphi}$$

where the phase difference is $\varphi=atan\frac{X_{LOAD}+X_{Th}}{R_{LOAD}+R_{Th}}$ and the impedance of the circuit is

$$z=\sqrt{\left(R_{LOAD}+R_{Th}\right)^2+\left(X_{LOAD}+X_{Th}\right)^2}\ .$$

The active power reaching the load is:

$$P=V_{Th}.I.\cos\varphi=\frac{V_{Th}^2}{\sqrt{\left(R_{LOAD}+R_{Th}\right)^2+\left(X_{LOAD}+X_{Th}\right)^2}}\cos\varphi$$

As can be seen from the above equation the two requirements to have maximal power are:

- $\cos\varphi$ should have a maximum. This happens when:

$$\cos\varphi=1\rightarrow atan\frac{X_{LOAD}+X_{Th}}{R_{LOAD}+R_{Th}}=0\rightarrow X_{LOAD}=-X_{Th}$$

- The denominator should have a minimum:

$$\sqrt{\left(R_{LOAD}+R_{Th}\right)^2+\left(X_{LOAD}+X_{Th}\right)^2}=MIN$$

If the reactive part of the denominator is removed ($X_{LOAD}=-X_{Th}$ the denominator becomes lowest. Considering this is also a requirement for the maximum of $\cos\varphi=1$ we apply it and obtain a purely resistive denominator:

$$\sqrt{\left(R_{LOAD}+R_{Th}\right)^2+\left(X_{LOAD}+X_{Th}\right)^2}=\sqrt{\left(R_{LOAD}+R_{Th}\right)^2}=R_{LOAD}+R_{Th}$$

We have already proven for DC circuit that for resistive circuits the power transfer is maximal if $R_{LOAD}=R_{Th}$. Then the active power reaching the load is maximal when the complex load impedance equals the complex conjugate of the Thevenin's complex impedance:

$$\dot{Z}_{LOAD}=\overset{*}{Z}_{Th}$$

5.4. Analysis of circuits in sinusoidal steady state

5.4.1. Equivallent complex impedance

All the rules shown for resistive DC circuits apply for analysis of circuits in AC state. Consider the circuit presented in fig. 5.28. The KVL is:

$$\dot{V}=\dot{V}_1+\dot{V}_2+\dot{V}_3=\left(Z_1+Z_2+Z_3\right)\dot{I}=Z_{\mathrm{IN}}\dot{I}$$

In other words the series complex impedance is:

$$Z_{\mathrm{IN}}=\sum Z_k$$

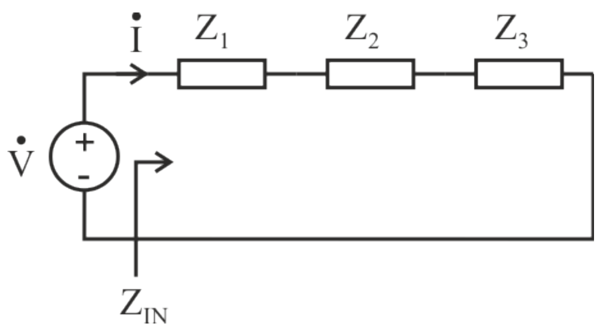


Fig. 5.28.

Next consider the circuit in fig. 5.29. The KCL could be written as:

$$\dot{I}=\dot{I}_1+\dot{I}_2+\dot{I}_3=\frac{\dot{V}}{Z_1}+\frac{\dot{V}}{Z_2}+\frac{\dot{V}}{Z_3}=\dot{V}\left(\frac{1}{Z_1}+\frac{1}{Z_2}+\frac{1}{Z_3}\right)=\dot{V}\frac{1}{Z_{¿}}$$

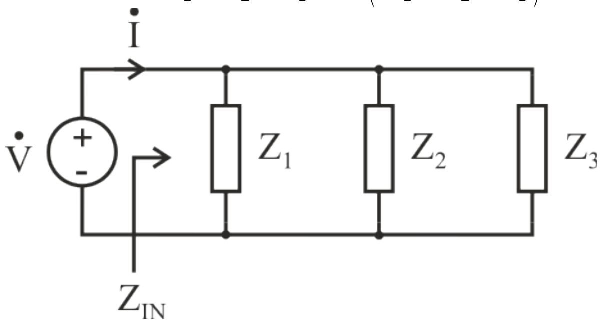


Fig. 5.29.

It can be seen that the equivalent parallel complex impedance is:

$$Z_{IN}=\frac{1}{\sum\frac{1}{Z_k}}=\frac{1}{\sum Y_k}$$

where $Y_k=\frac{1}{Z_k}$ is the complex admittance of the k -th branch.

Example: Determine the input impedance for the circuit in fig. 5.30 if $\omega=100\,rad/s$.

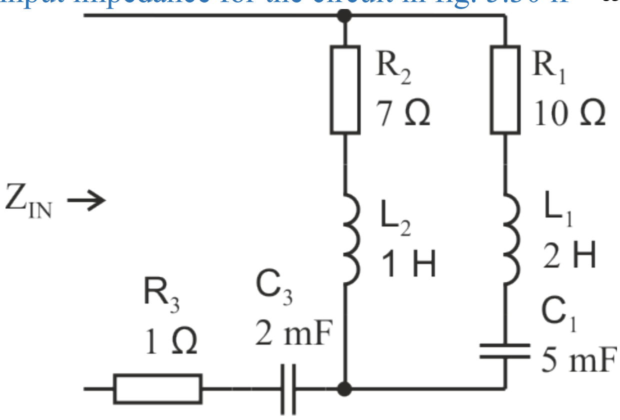


Fig. 5.30.

The capacitive and inductive reactances in the circuit are:

$$X_{L1}=\omega L_1=100\,\Omega$$
$$X_{L2}=\omega L_2=200\,\Omega$$
$$X_{C1}=\frac{1}{\omega C_1}=2\,\Omega$$
$$X_{C3}=\frac{1}{\omega C_3}=5\,\Omega$$

The complex impedances of the three branches are:

$$Z_1=R_1+jX_{L1}-jX_{C1}=10+j100-j2=10+j98\,\Omega$$
$$Z_2=R_2+jX_{L2}=7+j200\,\Omega$$
$$Z_3=R_3-jX_{C3}=1-j5\,\Omega$$

The input impedance of the circuit is:

$$Z_{IN}=\frac{Z_1Z_2}{Z_1+Z_2}+Z_3=\frac{(10+j98)(7+j200)}{10+j98+7+j200}+1-j5=6.26+j60.8\,\Omega$$

5.4.2. Circuit analysis

The analysis of circuits in sinusoidal steady state includes the following steps:

1. Transfer the circuit to the phasor domain;

2. Solve the problem using any of the DC circuit technics (Kirchhoff's laws, nodal analysis, mesh analysis, superposition, etc.);
3. Transfer the resulting phasor to the time domain.

The problem solving usually includes finding the currents and voltage drops and verifying the conservation of power.

5.4.2.1. Analysis using the Kirchhoff's laws

This is the most universal method because it could be applied for nonlinear circuits as well. This method includes writing a system of equations using the Kirchhoff's laws using the following rules:

- The number of equations in the system is equal to the number of the unknown currents in the circuit;
- The number of the KCL equations is equal to the number of nodes minus 1;
- The rest of the equations are according to the KVL.

Example: For the circuit in fig. 5.31 is known: $v(t)=5\sin(\omega t+45°)$, $i(t)=1.414\sin(\omega t)$, $R_1=2\Omega$, $\omega L_1=5\Omega$, $\frac{1}{\omega C_1}=5\Omega$, $R_2=3\Omega$, $\frac{1}{\omega C_2}=3\Omega$, $R_3=5\Omega$. Find the currents of the circuit and verify the conservation of power.

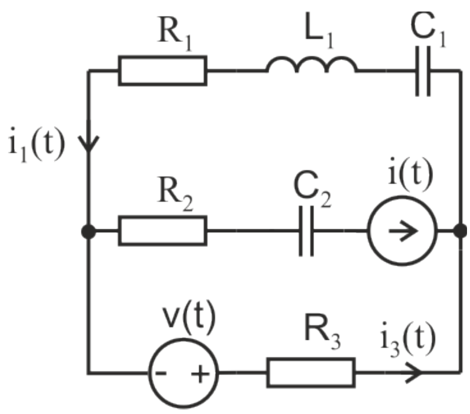


Fig. 5.31.

First we need to present the current and voltage source as complex effective values:

$$\dot{V}=\frac{5}{\sqrt{2}}e^{j45°}=3.54\,e^{j45°}=2.5+j2.5\,V$$

$$\dot{I}=\frac{1.414}{\sqrt{2}}e^{j0}=1\,A$$

There are 2 unknown currents so we need a system of two equations:

$$\left|\begin{array}{l}\dot{I}+\dot{I}_3=\dot{I}_1\\ \dot{V}=\dot{I}_3 Z_3+\dot{I}_1 Z_1\end{array}\right.$$

where $Z_1=R_1+j\left(\omega L_1-\frac{1}{\omega C_1}\right)=2+j(5-5)=2$ and $Z_3=R_3=5$.

The above system could be written in matrix form as:

$$\begin{bmatrix}\dot{I}_1\\ \dot{I}_3\end{bmatrix}\begin{bmatrix}1 & -1\\ 2 & 5\end{bmatrix}=\begin{bmatrix}1\\ 2.5+j2.5\end{bmatrix}$$

The determinants are:

$$\Delta=1*5+1*2=7$$

$$\Delta_1 = 5 + 2.5 + j2.5 = 7.5 + j2.5$$
$$\Delta_3 = 2.5 + j2.5 - 2 = 0.5 + j2.5$$

The solution is:

$$\dot{I}_1 = \frac{\Delta_1}{\Delta} = \frac{7.5 + j2.5}{7} = 1.07 + j0.36 = 1.13 e^{j18.6°}$$
$$\dot{I}_3 = \frac{\Delta_3}{\Delta} = \frac{0.5 + j2.5}{7} = 0.07 + j0.36 = 0.37 e^{j79°}$$

In sinusoidal form the currents are:

$$i_1(t) = 1.6 \sin(\omega t + 18.6°)$$
$$i_3(t) = 0.52 \sin(\omega t + 79°)$$

Next we need to verify the conservation of power but first we need to find out the voltage drop $\dot{V}_I$ on the current source. We could do that with a KVL for a loop through the current source:

$$\dot{V} = \dot{I}_3 Z_3 + \dot{V}_I - \dot{I} Z_2$$

where $Z_2 = R_2 - j\frac{1}{\omega C_2} = 3 - j3$.

Then the voltage drop $\dot{V}_I$ is:

$$\dot{V}_I = \dot{V} + \dot{I} Z_2 - \dot{I}_3 Z_3 = 2.5 + j2.5 + 1.(3 - j3) - 5.(0.07 + j0.36) = 5.15 - j2.3$$

The power of the sources is:

$$\dot{S}_{SRC} = \dot{V} \overset{*}{\dot{I}_3} + \dot{V}_I \overset{*}{\dot{I}} = (2.5 + j2.5)(0.07 - j0.36) + (5.15 - j2.3)(1) =$$
$$= 0.175 - j0.9 + j0.175 + 0.9 + 5.15 - j2.3 = 6.225 - j3.025\, VA$$

The power of the consumers is:

$$\dot{S}_{CONS} = I_1^2 . Z_1 + I^2 . Z_2 + I_3^2 . Z_3 = 1.13^2 .2 + 1^2 .(3 - j3) + 0.37^2 .5 = 2.55 + 3 - j3 + 0.685 = 6.235 - j3\, VA$$

It can be seen that $\dot{S}_{SRC} \approx \dot{S}_{CONS}$. The results do not much perfectly because of round ups.

5.4.2.2. Nodal analysis

The nodal analysis is based on the Kirchhoff's current low. Since KCL is valid for phasors we can analyze the circuit by nodal analysis. One of the nodes is grounded and the unknowns are the relative voltages of the other nodes.

Example: Analyze the circuit in fig. 5.32 and estimate the current through the inductor using nodal analysis.

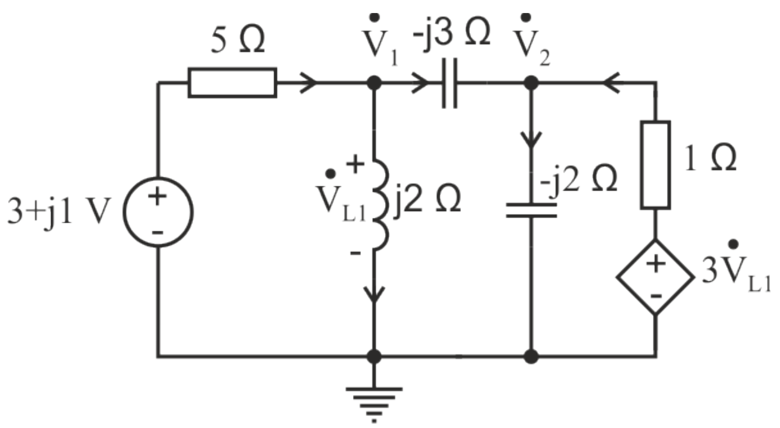


Fig. 5.32.

There are 3 nodes in the circuit and one of them is grounded. The other two nodal voltages are named $\dot{V}_1$ and $\dot{V}_2$. The dependent voltage source equals three times the inductor drop $\dot{V}_{L1}$

or:

$$3\dot{V}_{L1}=3\dot{V}_1$$

We write two equations based on the KCL for nodes 1 and 2:

$$\left|\begin{array}{l}\dfrac{3+j1-\dot{V}_1}{5}=\dfrac{\dot{V}_1-\dot{V}_2}{-j3}+\dfrac{\dot{V}_1}{j2}\\ \dfrac{\dot{V}_1-\dot{V}_2}{-j3}+\dfrac{3\dot{V}_1-\dot{V}_2}{1}=\dfrac{\dot{V}_2}{-j2}\end{array}\right. \rightarrow \left|\begin{array}{l}\dfrac{3+j1}{5}=\dot{V}_1\left(\dfrac{1}{5}+\dfrac{1}{j2}+\dfrac{1}{-j3}\right)+\dot{V}_2\dfrac{1}{j3}\\ 0=\dot{V}_1\left(\dfrac{1}{-j3}+3\right)+\dot{V}_2\left(\dfrac{1}{j3}-1+\dfrac{1}{j2}\right)\end{array}\right.$$

Then we write the it in matrix form and estimate the determinants:

$$\begin{bmatrix}\dot{V}_1\\ \dot{V}_2\end{bmatrix}\begin{bmatrix}0.2-j0.167 & j0.33\\ 3+j0.33 & -1-j0.83\end{bmatrix}=\begin{bmatrix}0.6+j0.2\\ 0\end{bmatrix}$$

$$\Delta=(0.2-j0.167)(-1-j0.83)-(j0.33)(3+j0.33)=-0.229-j0.99$$
$$\Delta_1=(0.6+j0.2)(-1-j0.83)=-0.434-j0.698$$
$$\Delta_2=-(0.6+j0.2)(3+j0.33)=-1.734-j0.798$$

Then the node voltages are:

$$\dot{V}_1=\frac{\Delta_1}{\Delta}=\frac{-0.434-j0.698}{-0.229-j0.99}=0.766-j0.261$$
$$\dot{V}_2=\frac{\Delta_2}{\Delta}=\frac{-1.734-j0.798}{-0.229-j0.99}=1.15-j1.49$$

And the complex current through the inductor is:

$$\dot{I}_{L1}=\frac{\dot{V}_1}{j2}=\frac{0.766-j0.261}{j2}=-0.131-j0.383$$

5.4.2.3. Mesh analysis

The meush analysis is based on the KVL and it should be used when there are many nodes. The idea is to create enough loops which will go through all the elements of the circuit.

Example: For the circuit in fig. 5.33 find the current $\dot{I}_{L2}$ using the mesh analysis method.

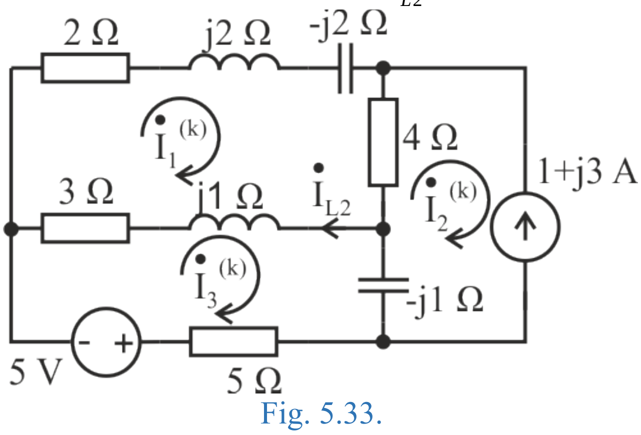


Fig. 5.33.

Since the mesh $\dot{I}_2^k$ goes through a current source with the opposite direction it is:

$$\dot{I}_2^k=-(1+j3)A$$

Then we write the KVL for the other mesh currents:

$$\left| \begin{aligned} 0 &= \dot{I}_1^k(2+j2-j2)+4\left(\dot{I}_1^k+1+j3\right)+(3+j1)\left(\dot{I}_2^k-\dot{I}_3^k\right) \\ -5 &= 5\dot{I}_3^k+(3+j1)\left(\dot{I}_3^k-\dot{I}_1^k\right)-j1\left(\dot{I}_3^k+1+j3\right) \end{aligned} \right.$$

or

$$\left| \begin{aligned} -4-j12 &= \dot{I}_1^k(2+4+3+j1)-\dot{I}_3^k(3+j1) \\ -5+j1-3 &= -\dot{I}_1^k(3+j1)+\dot{I}_3^k(5+3+j1-j1) \end{aligned} \right.$$

The above equations can be written in matrix form:

$$\begin{bmatrix} \dot{I}_1^k \\ \dot{I}_3^k \end{bmatrix} \begin{bmatrix} 9+j1 & -(3+j1) \\ -(3+j1) & 8 \end{bmatrix} = \begin{bmatrix} -4-j12 \\ -8+j1 \end{bmatrix}$$

The determinants are:

$$\Delta=72+j8-8-j6=64+j2$$
$$\Delta_1=(-4-j12)(8)+(3+j1)(-8+j1)=-57-j101$$
$$\Delta_3=(9+j1)(-8+j1)-(4+j12)(3+j1)=-73-j39$$

And the mesh currents are:

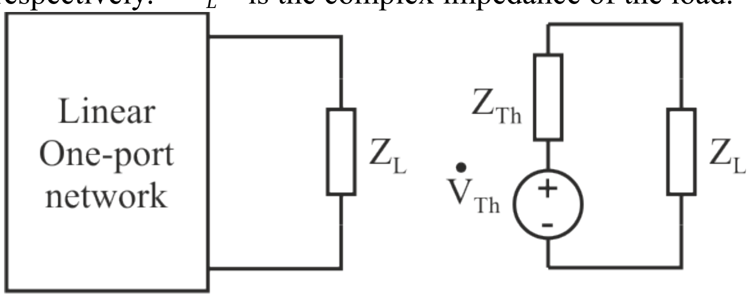
$$\dot{I}_1^k=\frac{\Delta_1}{\Delta}=\frac{-57+j101}{64+j2}=-0.939-j1.549$$
$$\dot{I}_3^k=\frac{\Delta_3}{\Delta}=\frac{-73+j39}{64+j2}=-1.159-j0.573$$

And the solution for $\dot{I}_{L2}$ is:

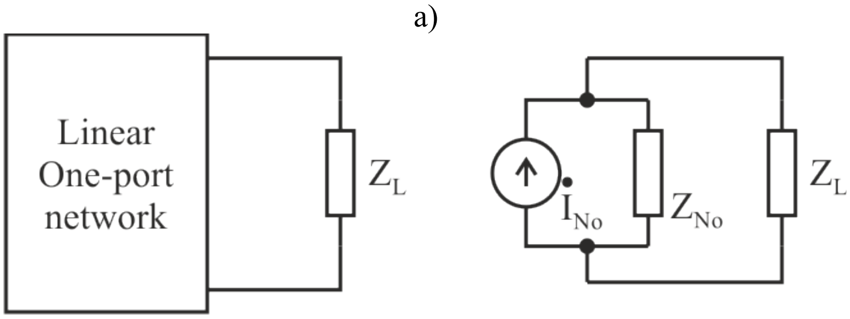
$$\dot{I}_{L2}=\dot{I}_1^k-\dot{I}_3^k=-0.939-j1.549+1.159+j0.573=0.22-j0.976$$

5.4.2.4. Thevenin and Norton equivallent circuits

Thevenin's and Norton's theorems could be applied for AC analysis as well. The equivallent Thevenin and Norton circuits are presented in fig. 5.34 where $Z_{Th}=Z_{No}$ is the equivallent complex impedance of the linear one-port and $\dot{V}_{Th}$ and $\dot{I}_{No}$ – the open circuit voltage and short circuit current respectively. Z_L is the complex impedance of the load.



a)



b)

Fig. 5.34. Thevenin (a) and Norton (b) equivalent circuits.

Example: For the circuit in fig. 5.35 estimate the current $\dot{I}_{L2}$ using the Thevenin equivallent circuit method.

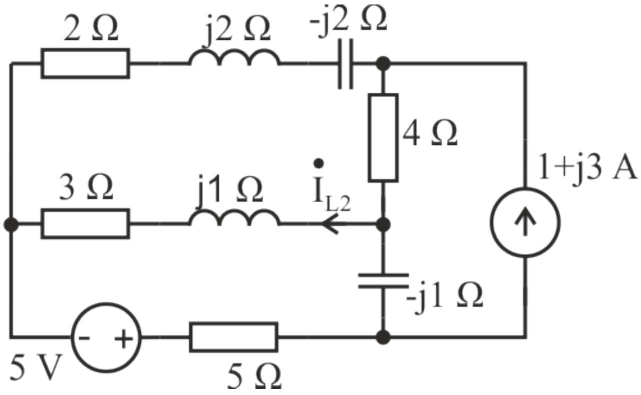


Fig. 5.35.

First we find the input complex impedance where the load was by replacing the votlage source with a short circuit and the current source with an open circuit (fig. 5.36):

$$Z_{Th}=Z_{ab}=\frac{(2+j2-j2+4)(5-j1)}{2+j2-j2+4+5-j1}=\frac{30-j8}{11-j1}=2.77-j0.48\,\Omega$$

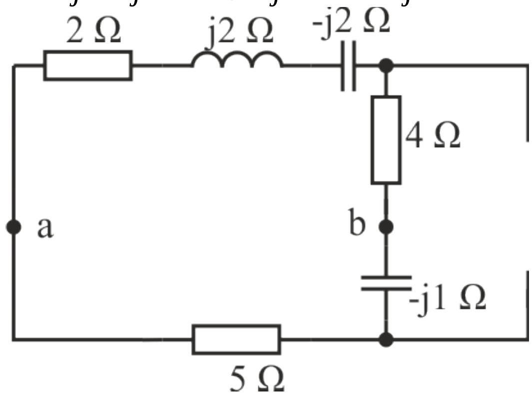


Fig. 5.36.

Next we need to find the open circuit voltage with the load removed (fig. 5.37):

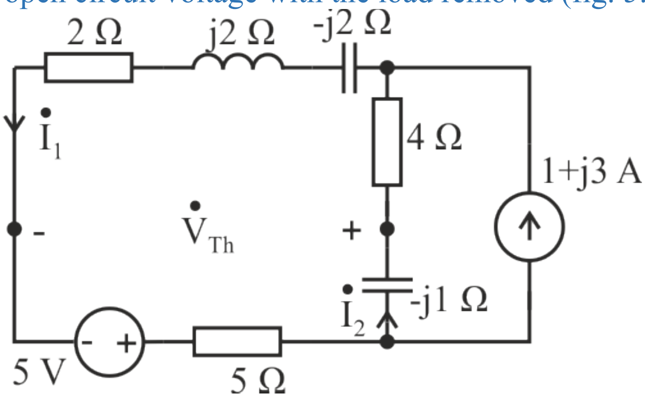


Fig. 5.37.

The KCL and KVL equations are:

$$\dot{I}_2+1+j3=\dot{I}_1$$

$$5=\dot{I}_1(5+2+j2-j2)+\dot{I}_2(4-j1)$$

In matrix form the equations become:

$$\begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 7 & 4-j1 \end{bmatrix} = \begin{bmatrix} 1+j3 \\ 5 \end{bmatrix}$$

The determinants are:

$$\Delta = 4 - j1 + 7 = 11 - j1$$
$$\Delta_1 = 4 - j1 + j12 + 3 + 5 = 12 + j11$$
$$\Delta_2 = 5 - 7 - j21 = -2 - j21$$

And the currents are:

$$\dot{I}_1 = \frac{\Delta_1}{\Delta} = \frac{12 + j11}{11 - j1} = 0.992 + j1.09$$
$$\dot{I}_2 = \frac{\Delta_2}{\Delta} = \frac{-2 + j21}{11 - j1} = -0.008 - j1.91$$

Then the open circuit voltage is:

$$5 = (0.992 + j1.09)5 + (-0.008 - j1.91)(-j1) + \dot{V}_{Th}$$
$$\rightarrow \dot{V}_{Th} = 5 - 4.96 - j5.45 - 0.008j + 1.91 = 1.95 - j5.46$$

Then the load current in the equivallent Thevenin circuit is:

$$\dot{I}_{L2} = \frac{\dot{V}_{Th}}{Z_{Th} + Z_{Load}} = \frac{1.95 - j5.46}{2.77 - j0.48 + 3 + j1} = 0.25 - j0.97\ A$$

References

1. Alexander Ch., Sadiku M. Fundamentals of electric circuits. Fifth edition. McGraw-Hill. 2013. Chapter 9, 10, 11.

2. http://www.electronics-tutorials.ws/category/accircuits